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An Investigation into the Possibility of Numerical Ephemeris Extension for GPS

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DST-Group–RR–0443

ABSTRACT

We investigate the possibility of a purely numerical extrapolation of parameters to extend the period of validity of any given set of GPS ephemeris parameters. The sought-for benefit is a faster time to first fix for a user who has been away from GPS satellite visibility for some days, and who thus holds outdated ephemeris data. Such a numerical extrapolation is not guaranteed a priori to result in more accurate satellite position forecasting, and indeed we show that for the extrapolation schemes chosen in this report, it does not result in more accurate forecasting. It therefore will not give any statistical improvement to the time taken for the above user to obtain a position fix.

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Published by
Cyber and Electronic Warfare Division
Defence Science and Technology Group
PO Box 1500
Edinburgh, SA 5111, Australia
Telephone: 1300 333 362
Facsimile: (08) 7389 6567

AR–016–954
September 2017

APPROVED FOR PUBLIC RELEASE

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Executive Summary

An entity that starts to acquire visibility of GPS satellites after a down time of some days might wish to establish its absolute position quickly. A user of GPS for this purpose needs accurate knowledge of the current positions of GPS satellites. These positions are predicted by the GPS receiver from its on-board ephemeris data. This data would normally be updated every two hours from satellite transmissions received if the GPS receiver were switched on and constantly visible to satellites. But a user who has been out of such visibility for some days holds old ephemeris data, and this data might no longer be an accurate parametrisation of the satellite orbits. Satellite positions predicted from such an ephemeris will tend to be inaccurate, resulting in a longer "time to first fix" for the receiver to lock on to a selection of satellites. This time might be unacceptably long.

In this report we investigate the extent to which such a user can predict satellite positions accurately from old ephemeris data, thus shortening the time to first fix. We examine the possibility of numerically extrapolating the outdated ephemeris data forward to the moment when the user sets his GPS receiver to begin searching for satellites, using that extrapolated data as a pseudo-current ephemeris. An improvement on this procedure based on physical modelling of satellite orbits is currently being investigated by commercial GPS receiver companies wanting to provide their civilian customers with shorter times to first fix in urban and indoor environments. We ask instead whether it's possible to update or "extend" an ephemeris purely by numerical extrapolation only. Our conclusion is that at least for the fairly generic extrapolation schemes chosen in this report, numerical extrapolation is not accurate enough to form a useful method of ephemeris extension.

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Don Koks completed a doctorate in mathematical physics at Adelaide University in 1996, with a thesis describing the use of quantum statistical methods to analyse decoherence, entropy and thermal radiance in both the early universe and black hole theory. He holds a Bachelor of Science from the University of Auckland in pure and applied mathematics and physics, and a Master of Science in physics from the same university with a thesis in applied accelerator physics (proton-induced X ray and γ ray emission for trace element analysis). He has worked on the accelerator mass spectrometry programme at the Australian National University in Canberra, and in commercial Internet development.

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1 Introduction

A standard GPS receiver that sees frequent use will contain comparatively up-to-date data describing satellite orbits. This data allows the receiver to predict approximate bearings and elevations of GPS satellites based on its last known position. These approximate look directions allow a quick lock on to whichever satellites are currently visible (a short "time to first fix"), so that the receiver can begin to download the necessary data strings that it uses to determine the satellites' ranges.

Up-to-date orbital ephemeris data is transmitted by the satellites every two hours, so if the receiver is continuously in view of satellites, it will always hold a current ephemeris in memory. But most receivers spend some or even most of their time switched off or else out of satellite visibility, resulting in ephemeris data that can be several days old when the receiver is called back into use. GPS satellite orbits are affected by various factors that render their ephemeris parameters "stale" after some hours or days, meaning those parameters cease to describe the satellite orbits very accurately. Satellite positions predicted from a stale ephemeris can be inaccurate, and this will lengthen the time to first fix; perhaps unacceptably so.

In this report we investigate a particular method that the receiver might implement to calculate satellite positions accurately from old ephemeris data, thus shortening its time to first fix. This method involves extrapolating the outdated ephemeris data forward to the moment of setting the receiver to search for satellites, then using that extrapolated data as a pseudo-current ephemeris. This procedure is called *ephemeris extension*, and variants of it are currently being investigated by commercial GPS receiver companies wanting to provide their civilian customers with shorter times to first fix in urban and indoor environments [1].

A note of terminology is in order here. By "extend" we mean running an algorithm that takes an old ephemeris and produces a pseudo-current one. By "predict" we mean taking an ephemeris and running the standard modified Kepler-equation algorithm that produces a satellite position. In a world with no perturbations so that orbits never changed, a single ephemeris would be current forever, and we could use it to predict arbitrarily far ahead with perfect accuracy. So we wish to compare two schemes: (a), extending an ephemeris n days ahead, then using the result as a pseudo-current ephemeris from which we predict zero time ahead to produce a satellite position, versus (b) predicting n days ahead from an old ephemeris to produce a satellite position.

Two choices exist to extend an ephemeris. The first models the physics that applies to the multi-body system comprising the oblate Earth with its rapidly changing tides, the Moon, Sun, and satellite. (Having said that, given that GPS satellites are certainly not in low-Earth orbits, knowledge of tides might not be necessary.) Reference [2] discusses the basic differential equation to be solved to propagate an orbit forward in time based on the necessary physical influences, and mentions several software packages that do this.

Useful details of the physics being modelled in this first choice are difficult or impossible to extract from published papers. For example, reference [1], also cited in [2], refers to the influence of Earth's oblateness and the Moon and Sun, but gives no in-depth details. But it does refer to orbit modelling, and presents plots that show position errors of tens of metres after one day of such an ephemeris extension. Reference [3] assumes an axially symmetric gravity field for Earth (which thus excludes ocean tide effects), but doesn't mention interactions with the Moon and Sun. It arrives at what appear to be satellite position errors of the order of 1 km after one day's extension, using a differential equation solver called the "method of averaging" that may have originated with Lagrange. The paper compares these position errors with corresponding values of perhaps 2 km after the same time produced from a conventional Runge-Kutta approach to solving its differential equations. Compare these values with position errors that are to be expected from using standard GPS ephemerides: predicting one day ahead (as opposed to *extending* one day ahead and then predicting zero time ahead from this extension) gives a position error of about 1 km, and predicting two days ahead gives a position error of about 2 km.

These results of 1 km and 2 km point to the main theme of this report, that *extending* an ephemeris n days ahead and using the result as a pseudo-current ephemeris gives similar results to *predicting* n days ahead from an old ephemeris.

Reference [4] seems to do much better than the above results. It mentions an axially symmetric model of Earth's gravity field, together with perturbations due to the Moon and Sun, and discusses the importance of solar radiation pressure on the satellite. It gives no calculations, and simply states that satellite position errors of better than 60 metres are obtainable after 3 days' extension.

The second choice for how ephemeris data might be extended is to analyse the variations in the ephemeris parameters themselves, with no reference to the physics. This might be the simpler approach to take, in that the parameters already incorporate the relevant physics, so that extrapolating them ahead in time might be treatable by mathematical methods alone. Reference [1] also discusses this approach, but it seems to relate to several perturbations from the Sun and Moon, so the actual approach used is not clear. That paper arrives at horizontal position-error values of tens of metres for the receiver after one day's extension, but the extent to which its approach has incorporated physical modelling is not clear.

1.1 Ephemeris Extension Does Not Guarantee Better Accuracy than Predicting

The key point to bear in mind in the following analysis is that in a fictional world where satellite orbits were never perturbed from their simple "keplerian" elliptical shapes, ephemeris data would never go stale. One ephemeris, never needing updating, would suffice for all GPS requirements. But a new ephemeris could certainly be produced every two hours, since some orbital parameters do change with time. Nevertheless, any ephemeris would predict the same satellite position as any other ephemeris, because although some parameters change with time, a parameter value that changes from one ephemeris to the next is offset by a change in time interval over which the prediction is made.[1] In this idealised world, extending an ephemeris by extrapolating its parameters over time would not be guaranteed to give an accurate position of the satellite, because pure numerical extrapolation is never dependable. That is, extending a set of ephemerides from a week ago with a sophisticated extrapolation technique would be inferior to using a single 20-year-old ephemeris. So even in our real, non-idealised world in which satellite orbits *are* perturbed by such things as the Sun and Moon, we should not assume that ephemeris extension is a-priori more accurate than prediction using an old ephemeris.

[1]To make the point with a simple analogy, suppose we wish to "predict" Salvador Dali's age in 1954, given his age in some initial year. Given his birth year of 1904 (age 0), we might predict 50 years ahead to arrive at age 50; or, given that he was 40 in 1944, we might predict 10 years ahead from 1944 to arrive again at age 50. His "age parameter" increases by one each year, but all choices of it must lead to the same age of 50 in 1954. So any one choice of that age parameter is as valid as any other choice.

2 How Accurately Must We Calculate a Satellite's Position?

To determine the accuracy to which ephemeris data must be known in order to establish a receiver's position to some given accuracy, we begin by examining a representative algorithm with which the GPS receiver can find its own position from transmissions received from the GPS satellites. Information on how any given GPS receiver determines its position is generally not available, but the following algorithm is representative of descriptions in GPS textbooks.

Suppose the receiver's position vector relative to Earth's centre is $\boldsymbol{R}$, to be determined. Place n satellites at positions relative to Earth's centre of $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$. These positions are assumed known to some extent; just how accurately the receiver requires to know them is what we will now establish.

Satellite i broadcasts a signal that the receiver receives at time t_i in the "Earth-Centred Inertial frame" (ECI). Being inertial, this frame admits a global time, and this is the time t used in standard GPS calculations. We assume this transmission time is known accurately, being encoded in the satellite's transmitted message.

The receiver receives this signal at a time that its possibly inaccurate receiver clock displays as t. This corresponds to an actual ECI time of $t + T$, where T is an unknown offset. The true range to the satellite is then $c(t + T - t_i)$ where c is the speed of light.[2]

The true range to a satellite is typically about 20,000 km, which light traverses in a time of about 20,000 km/(300,000 km/s), or 67 milliseconds. Earth turns through one revolution in the ECI in one sidereal day (about 23 hours, 56 minutes, 4 seconds, or 86,164 seconds), and given Earth's radius of 6.37×10^6 metres, it follows that in this short time of 67 milliseconds, a receiver on Earth's equator moves through a distance in the ECI of

$$\frac{0.067\,\text{s}}{86{,}164\,\text{s}} \times 2\pi \times 6.37 \times 10^6 \text{ metres} \simeq 31 \text{ metres}. \tag{2.1}$$

The GPS system takes this distance into account; but for our purposes we can ignore it, so will assume that for the duration of the satellite–receiver interaction, Earth is at rest. So, although the above inertial-frame analysis really only applies in the ECI, we'll assume it's happening in the "Earth-Centred Earth-Fixed (non-inertial) frame" (ECEF).[3] We'll use the standard cartesian coordinates of the ECEF in what follows.

[2]Contrast this true range with the satellite's *pseudo range*, given by $c(t - t_i)$, which is not needed in this report.

[3]In particular, it's impossible to define a global time coordinate in the rotating frame of the ECEF: relativity places a demand on the notion of simultaneity that cannot be achieved in a rotating frame. The time routinely used by GPS receivers is the time of the ECI, which is inertial and so does support a global time coordinate. So when considering very high timing accuracy in the ECEF, using GPS time is problematic, because it doesn't conform to the concept of a true time as used in relativity. For example, the passage of time indicated by a GPS receiver strapped to an atomic clock won't generally match the time elapsed ("proper time") as recorded by the atomic clock. GPS is usually described as "having relativity built into its clocks", because the satellite clocks are designed to tick slightly slowly in the factory; the tick rate then increases to the required value when the satellite is in orbit, due to the relativistic increase in the rate of flow of time experienced by the orbiting satellite. But that time speed-up relates to the ECI, in which GPS calculations are carried out. The bottom line is that the time used in GPS applies to the ECI and not the ECEF, even though it's routinely used in the ECEF of our everyday world. If our society's timing requirements continue to increase, this approximation of ECEF time by ECI time might become problematic.

For satellite i we have $|\boldsymbol{R} - \boldsymbol{r}_i| = c(t + T - t_i)$, where only $\boldsymbol{R}$ and T are unknown. Working in ECEF coordinates, write

$$\boldsymbol{R} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \boldsymbol{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}. \tag{2.2}$$

The two unknowns $\boldsymbol{R}$ and T thus comprise four unknown numbers X, Y, Z, T, so can be determined given t and the positions $\boldsymbol{r}_i$ and transmission times t_i of at least 4 satellites. In practice it's perhaps slightly easier to deal with squared distances (and taking square roots can be expensive computationally), so for satellite i write

$$\begin{aligned} f_i(\boldsymbol{R}, T) &\equiv |\boldsymbol{R} - \boldsymbol{r}_i|^2 - c^2(t + T - t_i)^2 \\ &= (X - x_i)^2 + (Y - y_i)^2 + (Z - z_i)^2 - c^2(t + T - t_i)^2 \,, \end{aligned} \tag{2.3}$$

and this is required to equal zero for each satellite. Consider a first-order Taylor expansion of $f_i(\boldsymbol{R}, T)$ around initial estimates $\boldsymbol{R}_0 = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$ and T_0:

$$f_i(\boldsymbol{R}, T) \simeq f_i(\boldsymbol{R}_0, T_0) + \nabla f_i(\boldsymbol{R}_0, T_0) \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \\ T - T_0 \end{bmatrix} \simeq 0 \,, \tag{2.4}$$

where the gradient $\nabla f_i(\boldsymbol{R}_0, T_0)$ is being written as a row matrix

$$\begin{aligned} \nabla f_i(\boldsymbol{R}_0, T_0) &= \left[\frac{\partial f_i}{\partial X}, \frac{\partial f_i}{\partial Y}, \frac{\partial f_i}{\partial Z}, \frac{\partial f_i}{\partial T}\right](X_0, Y_0, Z_0, T_0) \\ &= 2\left[X_0 - x_i, \;\; Y_0 - y_i, \;\; Z_0 - z_i, \;\; -c^2(t + T_0 - t_i)\right]. \end{aligned} \tag{2.5}$$

It follows from (2.4) that for n satellites,

$$\begin{bmatrix} f_1(\boldsymbol{R}, T) \\ \vdots \\ f_n(\boldsymbol{R}, T) \end{bmatrix} \simeq \begin{bmatrix} f_1(\boldsymbol{R}_0, T_0) \\ \vdots \\ f_n(\boldsymbol{R}_0, T_0) \end{bmatrix} + \begin{bmatrix} \nabla f_1(\boldsymbol{R}_0, T_0) \\ \vdots \\ \nabla f_n(\boldsymbol{R}_0, T_0) \end{bmatrix} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \\ T - T_0 \end{bmatrix}. \tag{2.6}$$

Equating the left-hand side of (2.6) with zero and solving for X, Y, Z, T in a least-squares sense gives these that least-squares estimates as

$$\begin{bmatrix} X \\ Y \\ Z \\ T \end{bmatrix} \simeq \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ T_0 \end{bmatrix} - \begin{bmatrix} \nabla f_1(\boldsymbol{R}_0, T_0) \\ \vdots \\ \nabla f_n(\boldsymbol{R}_0, T_0) \end{bmatrix}^{\#} \begin{bmatrix} f_1(\boldsymbol{R}_0, T_0) \\ \vdots \\ f_n(\boldsymbol{R}_0, T_0) \end{bmatrix}, \tag{2.7}$$

where "#" denotes the left pseudo inverse. For a matrix M where $M^t M$ isn't close to being singular (where "t" denotes the transpose), a pseudo inverse of $M^{\#} = (M^t M)^{-1} M^t$ can be used. But the pseudo inverse is not unique, and when M is less well behaved, other choices of the pseudo inverse give a more robust algorithm. For example, to implement a multiplication $M^{\#} v$ in Matlab code, we could try any of `inv(M'*M)*M'*v`, or `pinv(M)*v` (more robust to near-singular data), or `M\v` (even more robust to near-singular data).

Now, suppose the receiver has used a current ephemeris to calculate the satellite positions $\boldsymbol{r}_1$ to $\boldsymbol{r}_n$, its receiver records a time of sighting t, and the true times of broadcast $t_1, \ldots, t_n$ are also known, being written into the broadcast messages of the satellites (which have very good on-board clocks). Equation (2.7) defines an iterative procedure that begins with initial estimates $\boldsymbol{R}_0, T_0$ and refines them at each step, hopefully to converge to the true values $\boldsymbol{R}, T$. In practice we can begin with $\boldsymbol{R}_0$ set to latitude 0, longitude 0, height 0, and T_0 set to some value discussed shortly. The algorithm (2.7) is not expected to be completely well behaved, being based on a linearisation. It tends to fail when the first-time value of T_0 is less than the true value T. That might sound as if choosing a first-time value of T_0 will be problematic, as it should be greater than T, whose value is being sought. But the algorithm turns out to be quite well behaved even for very large first-time values of T_0, such as 100 seconds.

With four unknowns, the algorithm requires measurements from at least four satellites. We code it in Matlab and run it a large number of times for various random choices of well-spaced satellite positions (say, the satellites being a minimum of 10,000 km apart and all visible to the receiver), a receiver clock error T randomly chosen close to 1 s, and randomly chosen receiver positions that are all at heights ranging from -1 km to 9 km from Earth's surface. We find that in the absence of noise it converges to high accuracy very quickly. A typical set of successive distances between the latest position estimate and the new position estimate is the following (quoting two significant figures here): the first difference is about 2,300,000 kilometres; this halves at each iteration for the next 8 iterations to 7500 km, after which it drops more quickly, first to 2500 km, then 340 km, then 6.6 km, then 0.0025 km, then 3.5×10^{-10} km. At this stage, the new position estimate is within 10^{-10} km of the ground truth, and the estimated clock error differs from the true clock error by about 10^{-16} seconds. (Note that these numbers only express the mathematical convergence of the algorithm; they should not be interpreted as quantifying the physical accuracy that is being achieved.)

In fact, the algorithm might fail to converge once in every thousand runs, although this number varies wildly in actual simulations. Increasing the number of satellites to 5 increases the robustness of the algorithm significantly, and then it might not fail at all in 10,000 runs.

Suppose we now simulate using stale ephemeris data by adding down-range errors to the four satellite positions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4$. The errors are each chosen as normally distributed around 20 km with a standard deviation of 10 km. Running the algorithm shows that the estimates of our position are now typically in error by some tens of kilometres, while the error in the estimate of T is typically tens of microseconds. This position difference is, on average, about equally shared between height error and "projection-on-ground" error. If we now replace each down-range error with a cross-range error of the same typical magnitude and in a random transverse direction to the direction of the satellite, we find that the estimates of the receiver's position are now typically in error by perhaps 0.1 km, while the error in the estimate of T is negligible.

This work indicates that we must focus on estimating the satellites' down-ranges from the receiver. Estimating their cross-ranges is of much lesser importance.

3 Description of the Ephemeris Parameters

Here we describe the ephemeris parameters used by the receiver to calculate current positions of the GPS satellites. We require to examine the sensitivity of a satellite's predicted position to changes in each parameter, to determine the necessity of projecting this parameter ahead by the number of days that a receiver has been out of satellite view. If the satellite's predicted position *is* sensitive to changes in some parameter, we will have to project that parameter ahead in time by several days from its last-known (now stale) value. If the satellite's predicted position is *not* sensitive to changes in that parameter, we will use the last-known value of that parameter for the satellite prediction.

Ephemeris parameters are sourced from "Rinex" files, a standard text format used for promulgating ephemerides. These files are available in ".Z" compressed format at the `cddis.gsfc.nasa.gov` web site. Each file holds one day of ephemerides for usually all satellites, updated every two hours. To retrieve the Rinex file for a 3-digit day[4] `ABC` in a 4-digit year `PQRS`, visit the web address (noting that the following addresses contain a zero before the `.RS`)

`ftp://cddis.gsfc.nasa.gov/gps/data/daily/PQRS/brdc/brdcABC0.RSn.Z`

To retrieve a range of day numbers `ABC` to `DEF` in that one year, visit

`ftp://cddis.gsfc.nasa.gov/gps/data/daily/PQRS/brdc/brdc[ABC-DEF]0.RSn.Z`

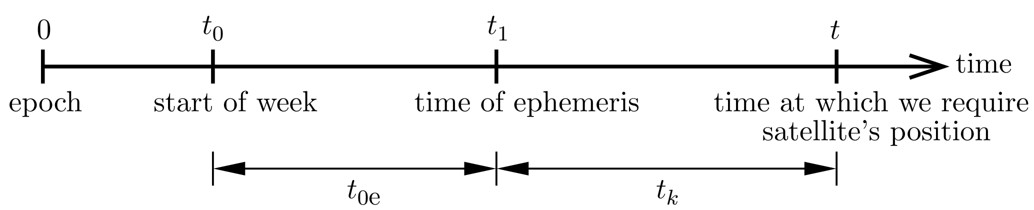
An explanation of some GPS nomenclature is necessary to explain how the ephemeris parameters must be used. First, all times in this report are given in UTC. (GPS time differs from UTC by the sum of a constant offset and some leap seconds. This time-dependent total offset has been accounted for in the calculations of this report.) Now:

- set t_0 to be the time at the "start of week" (midnight at the start of the most recent sunday), measured from a given epoch: any epoch will do, such as the start of the GPS era or, for that matter, some specified time in the year AD 1234;
- set t_1 to be the time at which the ephemeris is designed to be most accurate, measured from the same epoch, and
- set t to be the time at which the receiver requires to calculate the satellite's position, again measured from the same epoch.

Then the standard GPS engineering specification [5] uses the terms

$$t_{0e} = t_1 - t_0 \,, \quad t_k = t - t_1 \,. \tag{3.1}$$

The "time of ephemeris" is the time in the week at which the ephemeris is designed to apply most accurately. Only t_{0e} and t_k or their sum are used in calculations, and the fact that they are time *differences* is why the choice of the epoch of time zero is immaterial.



The GPS specification mixes references to the position of the ascending node as measured in the ECEF and the ECI, and so we require to relate these quantities. (Mixing reference frames gives no benefit here and can cause confusion, but it's simply what the specification does.) In particular, referring to Figure 1, we wish to estimate the terrestrial longitude Ω_{ECEF} of the ascending node at time t, given other parameters. Because no second derivatives

[4]E.g. day "002" denotes the second day of the year: 2[nd] January.

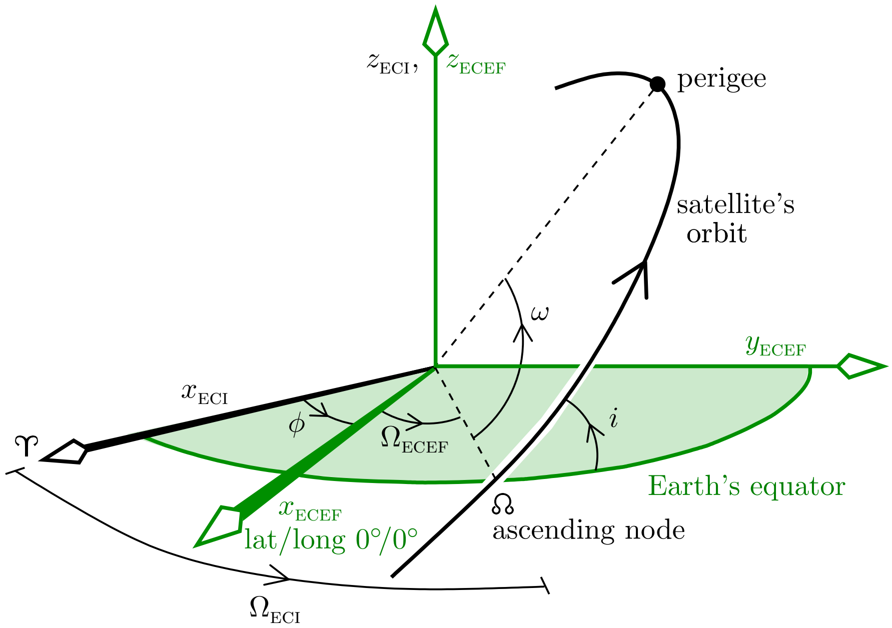


Figure 1: The parameters describing an orbit. Earth's spin axis defines the z axes of both the ECI and the ECEF. The x_{ECI} axis extends toward a point in the sky that is for our purposes fixed, the First Point of Aries ♈, defined in standard orbital theory using Earth's orbit about the Sun. The ascending node is commonly denoted ☊.

with respect to time exist in the specification, we assume that the rates of increase of the relevant parameters are constant enough throughout the week that a first-order estimate (Taylor expansion) is sufficient; hence we can immediately write

$$\Omega_{ECEF}(t) \simeq \Omega_{ECEF}(t_0) + \Omega'_{ECEF}(t_0)(t - t_0)\,. \tag{3.2}$$

We could replace t_0 with t_1 in (3.2), but because we must eventually identify the inadequately described terms in the GPS specification's expression written below as (3.7), we introduce t_0 in order to have a t_{0e} to compare our result with that equation.

The GPS ephemeris uses $\Omega'_{ECI}(t_0)$ (the time is irrelevant since this derivative is assumed to be constant throughout the week), so relate this to $\Omega'_{ECEF}(t_0)$ by referring to Figure 1 to write

$$\Omega_{ECEF}(t) = \Omega_{ECI}(t) - \phi(t)\,, \tag{3.3}$$

where $\Omega_{ECI}(t)$ is the *celestial longitude* (also known as *right ascension*) of the ascending node, and $\phi(t)$ is the "Greenwich sidereal angle",[5] the angle through which Earth has turned past the First Point of Aries. This angle $\phi(t)$ grows from 0 to 360° in one sidereal day—whose length is, conventionally, 23 hours, 56 minutes, 4.0989036903511 seconds [6]. In that case

$$\Omega'_{ECEF}(t_0) = \Omega'_{ECI}(t_0) - \phi'(t_0)\,. \tag{3.4}$$

[5]The Greenwich sidereal angle is often called *sidereal time*, or even *Greenwich apparent sidereal time*. I think the historical name "sidereal time" is unfortunate because $\phi(t)$ is simply an angle, and is always used as an angle; the fact that the spinning Earth resembles a giant clock is not actually useful quantitatively. When referred to as sidereal time, the angle $\phi(t)$ is often specified in hours/minutes/seconds, where 24 such angular hours are defined to be exactly 360°. But note that one of these angular hours does not in any way correspond to one hour of time. Even when this "sidereal time" is specified in hours/minutes/seconds, it still denotes an angle, never a time.

Table 1: The relation of some GPS terms to our terminology. The meaning of Ω_0 is taken from [7].

This report:	$\Omega_{\text{ECEF}}(t)$	$\Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}}$	$\dot{\Omega}_{\text{ECI}}$	$\dot{\phi}$	$t - t_0$
GPS specification:	Ω_k	Ω_0	$\dot{\Omega}$	$\dot{\Omega}_e$	$t_{0\text{e}} + t_k$

Both derivatives on the right-hand side of (3.4) are treated in the GPS specification as constant throughout the week, so write $\Omega'_{\text{ECI}}(t_0)$ as $\dot{\Omega}_{\text{ECI}}$ and Earth's spin rate $\phi'(t_0)$ as $\dot{\phi}$. (Earth's spin rate $\phi'(t_0)$ is 2π radians in one sidereal day, or $7.2921151467 \times 10^{-5}$ rad/s, which is the WGS-84 standard value.) Equation (3.2) is now written as

$$\Omega_{\text{ECEF}}(t) \simeq \Omega_{\text{ECEF}}(t_0) + \big[\dot{\Omega}_{\text{ECI}} - \dot{\phi}\big](t - t_0)\,. \tag{3.5}$$

This equation is where the standard instructions for calculating satellite positions in Table 20-IV of the GPS specification could have stopped; but that table re-arranges the terms in (3.5) in an obscure fashion, as we'll see shortly. To make contact with that GPS terminology, refer to (3.1) to write $t - t_0$ as $t_{0\text{e}} + t_k$, so that (3.5) becomes

$$\begin{aligned} \Omega_{\text{ECEF}}(t) &\simeq \Omega_{\text{ECEF}}(t_0) + \big[\dot{\Omega}_{\text{ECI}} - \dot{\phi}\big](t_{0\text{e}} + t_k) \\ &= \Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}} - \dot{\phi}\, t_{0\text{e}} + \big[\dot{\Omega}_{\text{ECI}} - \dot{\phi}\big] t_k\,. \end{aligned} \tag{3.6}$$

The second line of (3.6) must be compared to an equation in Table 20-IV of the GPS specification, where we find the following expression labelled the "corrected longitude of the ascending node", which uses that specification's notation:

$$\Omega_k = \Omega_0 + \big[\dot{\Omega} - \dot{\Omega}_e\big] t_k - \dot{\Omega}_e t_{0\text{e}}\,. \tag{3.7}$$

Although the GPS specification gives a recipe for calculating a satellite's position, it lacks precise definitions of its ephemeris parameters, and hence some inference is necessary to ascertain the meanings of the symbols in (3.7). It's clear from Table 20-IV that "$\dot{\Omega}_e$" in (3.7) is equivalent to our $\dot{\phi}$. We then infer that "$\dot{\Omega}$" in (3.7) is equivalent to our $\dot{\Omega}_{\text{ECI}}$. The use of "$\Omega_k$" in the specification makes clear that this equals our $\Omega_{\text{ECEF}}(t)$. That means "Ω_0" must equal our $\Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}}$. This last identification is totally obscure, but is corroborated by reference [7], an established GPS text which makes exactly the same identification. So we conclude that the GPS specification uses the terms shown in our Table 1.

What is the physical meaning of the GPS specification's Ω_0? Given that it equals our $\Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}}$, use (3.1) and (3.3) to write

$$\begin{aligned} \Omega_0 \equiv \Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}} &= \Omega_{\text{ECI}}(t_0) - \phi(t_0) + \dot{\Omega}_{\text{ECI}}[t_1 - t_0] \\ &= \Omega_{\text{ECI}}(t_1) - \phi(t_0)\,. \end{aligned} \tag{3.8}$$

That is, Ω_0 is the celestial longitude of the ascending node at t_1 minus the celestial longitude of Greenwich at t_0. This subtraction has no physical or geometrical significance. The specification calls it the "longitude of the ascending node at weekly epoch" (where "weekly epoch" is presumably the start of week t_0), which is clearly wrong. Reference [7] agrees and describes Ω_0 as simply a mislabelled parameter in the GPS specification.

A description of the format of Rinex files is given in [8]. The use of upper and lower case in the parameter names in that document is haphazard, so here is a list of the relevant parameters as written in that document, together with the name or symbol by which they are referred to in this report. Only the parameters listed below are used for calculating a satellite's position. Some of these parameters were shown in Figure 1 on page 7.

`Cus, Cuc`: c_{us}, c_{uc} (radians), corrections to true anomaly.

`Crs, Crc`: c_{rs}, c_{rc} (metres), corrections to satellite–Earth distance.

`CIS, Cic`: c_{is}, c_{ic} (radians), corrections to orbital inclination.

`Toe`: $t_{0\text{e}}$ or $t_1 - t_0$ (seconds), time elapsed from start of week t_0 to time of ephemeris t_1.

`M0`: $M(t_1)$ (radians), mean anomaly M at time of ephemeris t_1.

`Delta n`: ΔN (radians/second), which is $M'(t_1) - 2\pi/\text{period}$. Throughout the week the mean anomaly M varies approximately linearly, with value

$$M(t) \simeq M(t_1) + (2\pi/T + \Delta N)(t - t_1)\,, \tag{3.9}$$

where T is the satellite's period. In a true keplerian orbit $M'(t) = 2\pi/T$ exactly, so ΔN is exactly zero for such an orbit.

`e`: e, the orbital eccentricity, treated as constant over the week.

`sqrt(A)` or $\sqrt{A}$: $\sqrt{a}$ (metres$^{1/2}$), square root of semi-major axis length, treated as constant over the week. The square root is specified to simplify computation, since we later require both a and $a^{3/2}$.

`OMEGA`: The GPS specification's Ω_0 (radians), being $\Omega_{\text{ECEF}}(t_0) + \dot{\Omega}_{\text{ECI}}\, t_{0\text{e}}$ or $\Omega_{\text{ECI}}(t_1) - \phi(t_0)$: *almost* the terrestrial longitude (i.e. measured in Earth's equatorial plane from Greenwich) of the ascending node at start of week t_0, but actually a quantity with no proper physical meaning. Discussed on the preceding page.

`OMEGA DOT`: $\dot{\Omega}_{\text{ECI}}$ or $\mathrm{d}\Omega_{\text{ECI}}/\mathrm{d}t$ (radians/second). Despite the name, this is *not* any rate of increase of "`OMEGA`". Instead, Ω_{ECI} is the *celestial* longitude of the ascending node—that is, measured in Earth's equatorial plane from the First Point of Aries. The rate of increase of Ω_{ECI} is treated as constant over the week.

`i0`: $i(t_1)$ (radians), orbital inclination relative to Earth's equatorial plane, at time of ephemeris t_1.

`IDOT`: $\mathrm{d}i/\mathrm{d}t$ (radians/second), treated as constant over the week.

`omega`: ω (radians), argument of perigee measured in orbit plane from ascending node (which itself is in Earth's equatorial plane). Treated as constant over the week.

4 Details of Calculating a Satellite's Position

The position of a satellite at a given time can be calculated using the steps in this section. My notation here follows standard orbital theory, and is self consistent and mathematically straightforward. It includes the non-keplerian perturbations of Table 20-IV in [5].

Although the algorithm below is entirely equivalent to that of Table 20-IV, it doesn't quite match that table's notation, which I find to be awkward and un-insightful. (In particular, I have replaced Table 20-IV's u with direct reference to the true anomaly θ.) The satellite positions predicted by the algorithm below differ by only a few centimetres from those calculated using the recipe in Table 20-IV. This tiny difference is due to numerical round-off errors in the two computations.

In principle, we might use the ephemeris of any convenient date to predict a satellite's position; but the older the ephemeris of a GPS satellite's non-keplerian orbit, the less reliable will be the predictions based on it. Once an ephemeris has been read from a Rinex file, we supply a UTC time at which the satellite's position is required, and this becomes the time t in Section 3. Our goal might be to calculate the ECEF coordinates of the vector pointing from a given place on Earth to the satellite; but the central task is to calculate the ECEF coordinates of the vector pointing from Earth's centre to the satellite, which is what we address in this report.

Begin by calculating the satellite's mean anomaly M at the requested time t. For this we require its period T, found from

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}\,, \tag{4.1}$$

where μ is the product of the gravitational constant and Earth's mass, set in the WGS-84 system to be $\mu = 3.986005 \times 10^{14}$ SI units.

Now evolve the mean anomaly from its value at t_1 up to the requested time t using (3.9) (treating that equation as an exact equality), then convert $M(t)$ to the eccentric anomaly $E(t)$ in the standard way of keplerian orbital theory by solving $M = E - e\sin E$ numerically. (This is easily accomplished by beginning with $E = 0$, then iterating with $E_{\text{new}} = M + e\sin E_{\text{old}}$, using radians. The iterations converge sufficiently in three or four steps.) Convert the eccentric anomaly $E(t)$ to the true anomaly $\theta(t)$ in the standard way by solving the following for θ:

$$\sin\theta = \frac{\sqrt{1-e^2}\,\sin E}{1-e\cos E}\,, \qquad \cos\theta = \frac{\cos E - e}{1 - e\cos E}\,. \tag{4.2}$$

The GPS specification now calculates non-keplerian corrections to

- the satellite's distance $r(t)$ from Earth's centre,
- the true anomaly that it writes as "ν_k", but which is called $\theta(t)$ here, to highlight the fact that r, θ are just Earth-centred polar coordinates in the orbital plane, and
- the orbital inclination $i(t)$.

In particular, the before-correction values of r and i are

$$r(t) = a[1 - e\cos E(t)]\,, \quad i(t) = i(t_1) + \mathrm{d}i/\mathrm{d}t\,(t - t_1)\,. \tag{4.3}$$

Corrections to these are calculated by first defining $\Phi(t) \equiv \theta(t) + \omega(t)$, then writing

$$\begin{aligned}
\delta\theta &= c_{us}\sin 2\Phi + c_{uc}\cos 2\Phi\,,\\
\delta r &= c_{rs}\sin 2\Phi + c_{rc}\cos 2\Phi\,,\\
\delta i &= c_{is}\sin 2\Phi + c_{ic}\cos 2\Phi\,.
\end{aligned} \tag{4.4}$$

The corrections are now applied as

$$\begin{aligned} \theta(t) &\to \theta(t) + \delta\theta\,, \\ r(t) &\to r(t) + \delta r\,, \\ i(t) &\to i(t) + \delta i\,. \end{aligned} \tag{4.5}$$

The true anomaly is in fact not necessary to calculate the satellite's position; the eccentric anomaly (which must always be calculated anyway) could be used instead—and virtually always is, in orbital mechanics. The above correction to the true anomaly $[\theta(t) \to \theta(t) + \delta\theta]$ could easily have been implemented in the GPS standard as a correction to the eccentric anomaly "$E \to E + \delta E$", thus removing the obligation of every receiver in the world to have to calculate the true anomaly, as they now must do.[6]

Define the orbital plane's "OP" coordinates: their origin is Earth's centre, the x_{OP} axis is the elliptical orbit's major axis through the perigee ($\theta = 0$), y_{OP} is the minor axis through $\theta = 90°$, and z_{OP} completes a right-handed set. The orbital-plane coordinates of the satellite's position vector $\boldsymbol{r}(t)$ (the vector from Earth's centre to the satellite) are then, using the true anomaly,

$$[\boldsymbol{r}(t)]_{\text{OP}} = \begin{bmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \\ 0 \end{bmatrix}. \tag{4.6}$$

We now calculate the terrestrial longitude of the ascending node, $\Omega_{\text{ECEF}}(t)$, referred to in Section 3. We could use (3.5) here, but we could just as well have derived (3.5) using the time of ephemeris t_1 in place of the start of week t_0:

$$\Omega_{\text{ECEF}}(t) \simeq \Omega_{\text{ECEF}}(t_1) + \big[\dot\Omega_{\text{ECI}} - \dot\phi\big](t - t_1)\,. \tag{4.7}$$

To be shown later in Figure 7, it will prove useful to use (4.7) in place of (3.5), because $\Omega_{\text{ECEF}}(t_1)$ will turn out to be more easily predictable than $\Omega_{\text{ECEF}}(t_0)$ when we come to consider ephemeris extension. Note that (3.5) and (4.7) give the same value for $\Omega_{\text{ECEF}}(t)$, because implicit in their construction is the approximation that $\Omega_{\text{ECEF}}(t)$ evolves at a constant rate. So we are free to choose any moment in time from which to predict $\Omega_{\text{ECEF}}(t)$.

Finally, convert orbital-plane coordinates to ECEF coordinates [9]:

$$[\boldsymbol{r}(t)]_{\text{ECEF}} = E_3(\Omega_{\text{ECEF}})\, E_1(i)\, E_3(\omega)\, [\boldsymbol{r}(t)]_{\text{OP}}\,, \tag{4.8}$$

where E_1 and E_3 are Euler matrices:

$$E_1(\alpha) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad E_3(\alpha) \equiv \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.9}$$

The above calculations will give the same results as the recipe given in Table 20-IV in [5], agreeing to the few centimetres referred to earlier.[7]

[6] Given that $\sqrt{a}$ is given in the ephemeris rather than a itself, it seems that computational efficiency was intended in the GPS standard; and yet the above calculation of the true anomaly is completely unnecessary for calculating a satellite's position. This seems to be an example of how a preliminary working specification can become set in stone.

[7] If implementing the recipe in Table 20-IV, take care to implement its expressions involving $\tan^{-1}$ and $\cos^{-1}$ functions in a four-quadrant way: if they are interpreted literally as written, they can give wrong results. In contrast, (4.2) above is fully correct.

When the above discussion is combined with, say, a several-day-old Rinex ephemeris to predict the bearings and elevations of all satellites visible at some moment, the resulting angles have been found to agree (on a selection of randomly chosen days) with those displayed by a hand-held Defence GPS receiver used on the Defence Science and Technology Group's site in Adelaide, up to the receiver's read-out accuracy. Admittedly this accuracy is not ultra fine, since the receiver rounds all displayed values to the nearest degree. Even so, mistakes or misinterpretations of the meaning of orbital data tend to produce predictions of bearing/elevation that are catastrophically wrong; so we conclude that the above interpretations of all parameters are correct.

Note that UTC times in this report are written in the format "year-month-day-hour-minute-second", and all calculations here that refer to UTC have all relevant leap seconds included when converting to elapsed times. For example, the number of seconds elapsed from 2015-6-30-23-59-59 to 2015-7-1-0-0-0 is exactly 2, due to the leap second that occurred then.

5 Time Evolution of Ephemeris Parameters

Below are shown representative plots of the time evolution of the ephemeris parameters involved in calculating a satellite's position. A year-long range of dates has been chosen as 2014-1-1 to 2014-12-31. The Rinex file for each day in 2014 holds an ephemeris for (usually) each satellite at approximately two-hour intervals, so it has roughly 12 instances of each ephemeris parameter. A typical entry in a Rinex file might be the following (note: the "D" in these lines denotes a power of ten):

```
17 15  4  2  7 59 44.0-0.164222437888D-03-0.204636307899D-11 0.000000000000D+00
    0.900000000000D+01 0.214687500000D+02 0.484127308703D-08-0.426604535688D+00
    0.122562050819D-05 0.101049217628D-01 0.395812094212D-05 0.515373473740D+04
    0.374384000000D+06-0.596046447754D-07-0.193541457015D+01 0.987201929092D-07
    0.971537974102D+00 0.313218750000D+03-0.201460626835D+01-0.848463913356D-08
    0.243224416987D-09 0.100000000000D+01 0.183800000000D+04 0.000000000000D+00
    0.200000000000D+01 0.000000000000D+00-0.107102096081D-07 0.900000000000D+01
    0.373956000000D+06 0.400000000000D+01 0.000000000000D+00 0.000000000000D+00
```

These eight lines contain the following fields, ordered as written. Some of the fields are not used in this report, but are named here for completeness.

PRN year month day hour minute second clockBias clockDrift clockDriftRate			
IODE	c_{rs}	ΔN	$M(t_1)$
c_{uc}	e	c_{us}	$\sqrt{a}$
t_{0e}	c_{ic}	Ω_0	c_{is}
$i(t_1)$	c_{rc}	ω	$\mathrm{d}\Omega_{\mathrm{ECI}}/\mathrm{d}t$
$\mathrm{d}i/\mathrm{d}t$	codesOnL2	gpsWeek	L2PDataFlag
accuracy	health	TGD	IODC
txTime	fitInterval	spare1	spare2

We read the 365 Rinex files and concatenate the data of all ephemerides, then plot the value of each ephemeris parameter over the whole of 2014. Each plot then has approximately 365 × 12 points. Figure 2 shows the six parameters c_{us} etc. listed in Section 3, for satellite "PRN 1".

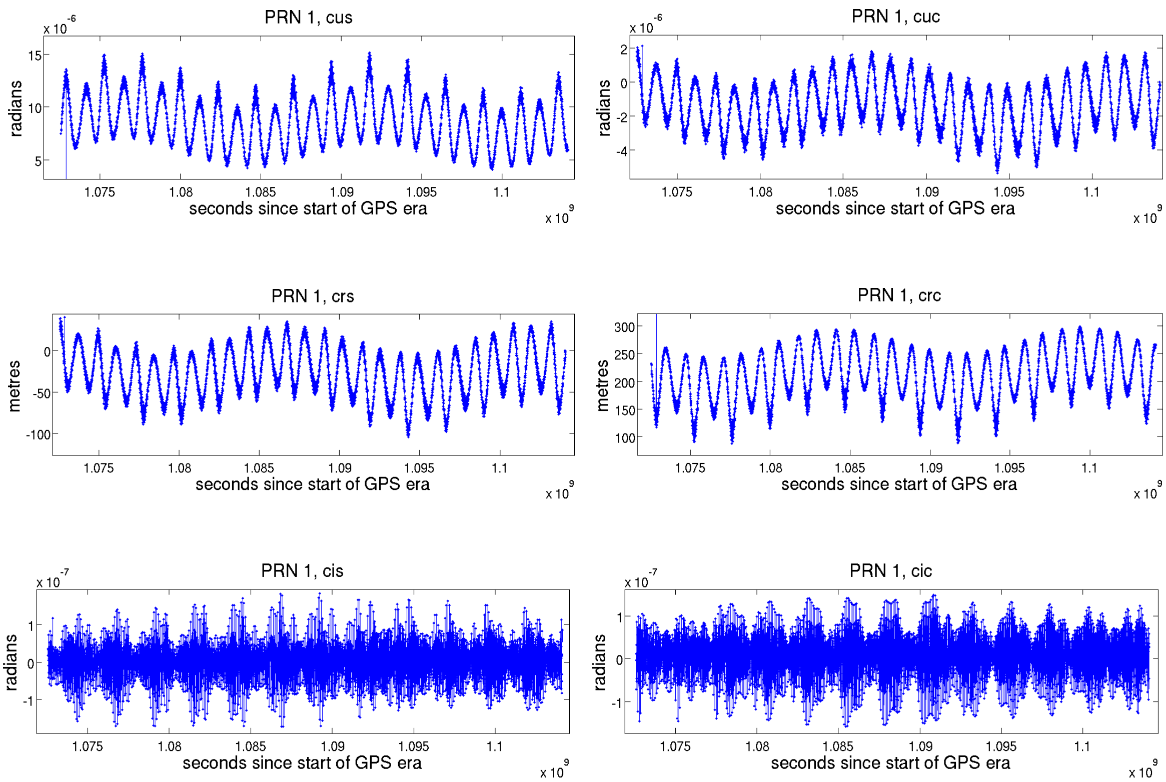


Figure 2: Variation of the parameters used to correct satellite range, true anomaly, and orbital inclination, over the year 2014. ***Top***: c_{us}, c_{uc}. ***Middle***: c_{rs}, c_{rc}. ***Bottom***: c_{is}, c_{ic}. *Note the outlier occurring early in the year. This complicates the automation of any ephemeris-extension algorithm.*

(The time axis is measured in seconds since the start of the GPS era, conventionally set at 1980-1-6-0-0-0.) Note the outlier near the start. This outlier is also clearly seen on the plot of eccentricity in Figure 3, and can be traced directly back to the Rinex file `brdc0040.14n` for 2014-1-4. The lines below come from that file for time 0:00:00, with the eccentricity the last number quoted:

```
 1 14  1  4  0  0  0.0 0.980868935585D-04 0.261479726760D-11 0.000000000000D+00
    0.740000000000D+02-0.155312500000D+02 0.423303346569D-08 0.213295779364D+01
   -0.735744833946D-06 0.261628220324D-02
```

The eccentricity has a normal-looking value of 0.2616×10^{-2}. Now in the same file at time 13:59:44 we find the outlier:

```
 1 14  1  4 13 59 44.0-0.156176742166D-03 0.795807864051D-12 0.000000000000D+00
    0.100000000000D+01 0.404375000000D+02 0.569809449136D-08 0.140912302660D+00
    0.213831663132D-05 0.645678443834D-02
```

where the eccentricity is now 0.6456×10^{-2}. This outlier might indicate a course correction of the satellite, since such spikes do appear in other ephemerides when a parameter changes value abruptly across the spike. In the present case there are no abrupt changes across the spike for the parameters where the spike occurs. As a side remark, the Rinex files do appear occasionally to have been edited at their source *after* they were first posted to the internet. This is evident when a Rinex file has been re-downloaded after some while and, when compared with the first version downloaded, is found to have had a single parameter changed.

Figure 3 shows the three parameters $M(t_1)$, ΔN, e. The mean anomaly has been "unwrapped", in the sense of applying the Matlab command `unwrap`, which undoes the "modulo 2π" behaviour of angles that are increasing or decreasing. This eliminates jumps from $-\pi$ to π in the plot, and hence brings out the true structure of the mean anomaly.

Unlike the other parameters, the mean anomaly shown in Figure 3 might appear to have no outlier, but examination of the Rinex file `brdc0040.14n` reveals what does appear to be an anomalous value; but because the value in the relevant field of the Rinex files is constrained to be between $-\pi$ and π, this anomalous value has caused no harm.

Figure 4 shows the two parameters $\sqrt{a(t_1)}$, $\omega(t_1)$.

Figure 5 shows the two parameters $i(t_1)$, $\mathrm{d}i/\mathrm{d}t$.

Figure 6 shows the two parameters $E(t_1)$, $\theta(t_1)$.

Figure 7 shows the three parameters $\Omega_{\text{ECEF}}(t_0)$, $\Omega_{\text{ECEF}}(t_1)$, $\dot{\Omega}_{\text{ECI}}$.

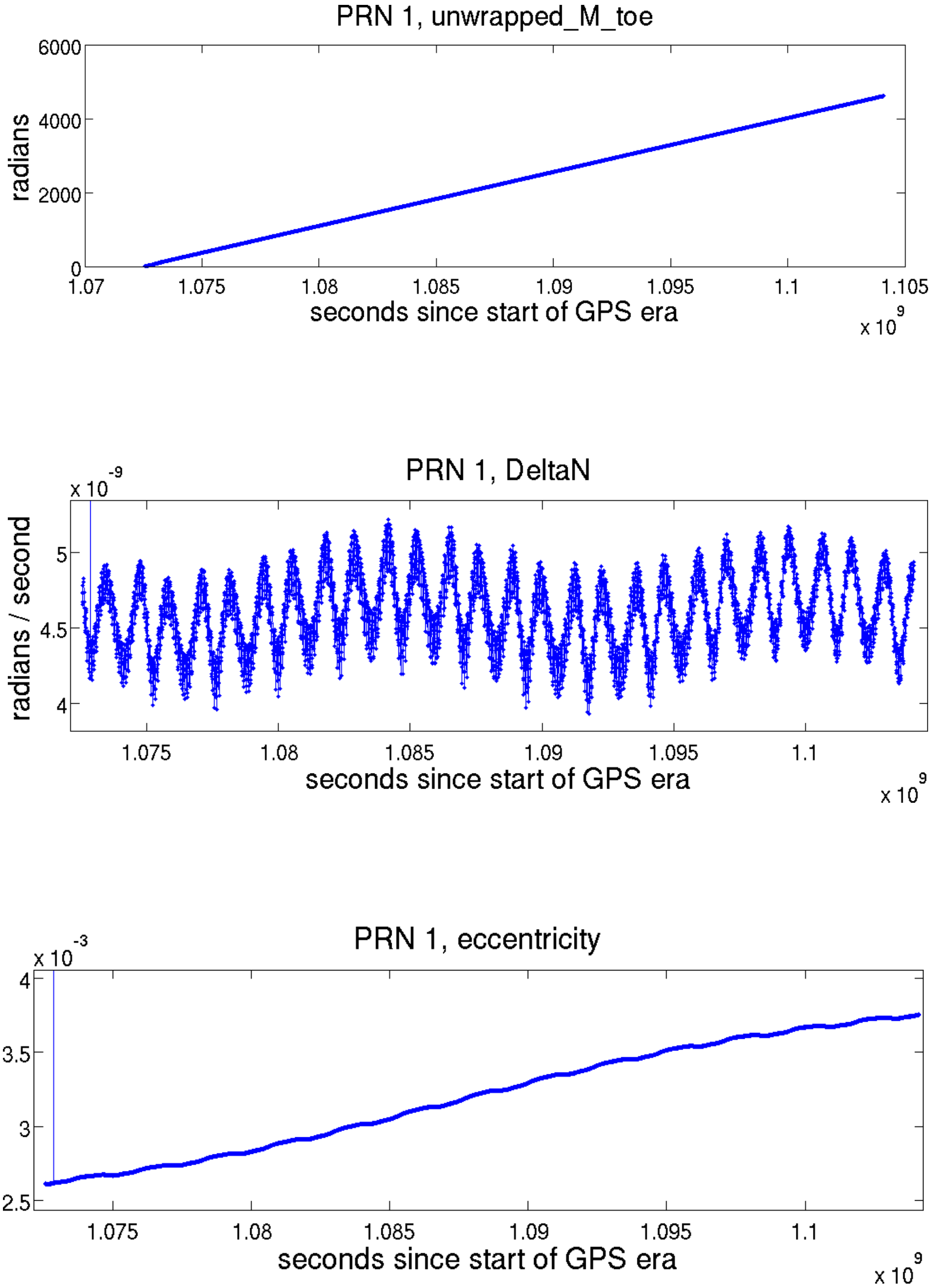


Figure 3: Variation of the mean anomaly $M(t_1)$ at time of ephemeris, the correction ΔN to its rate of increase, and the eccentricity e, over the year 2014

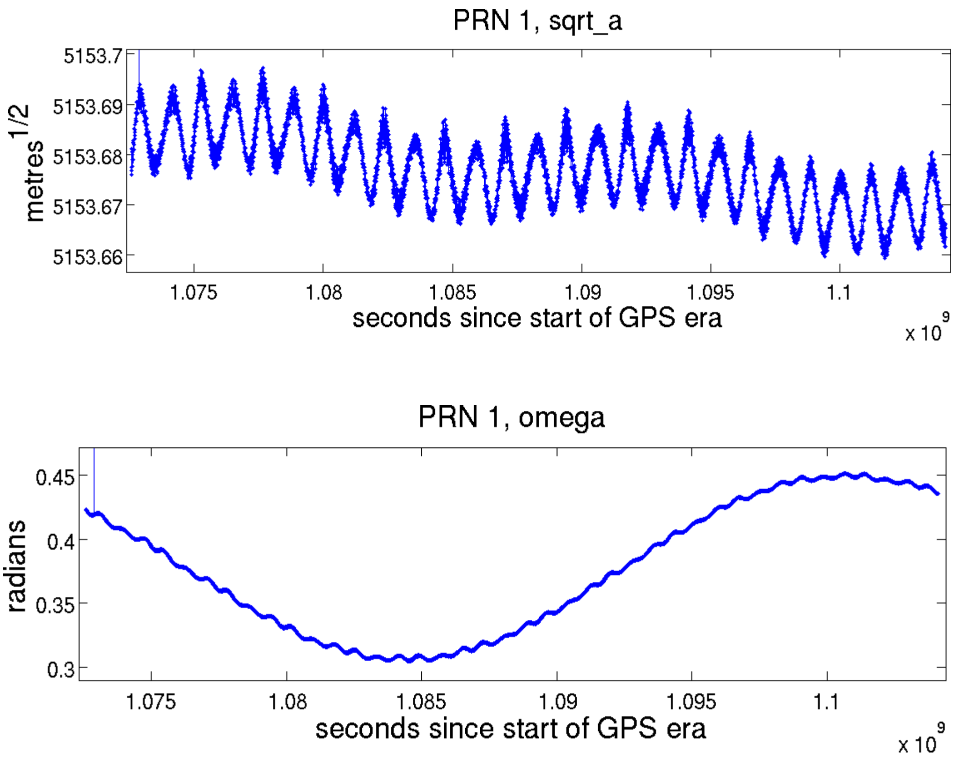


Figure 4: Variation of the square root of the semi-major axis length $\sqrt{a}$ *and the argument of perigee* ω *over the year 2014*

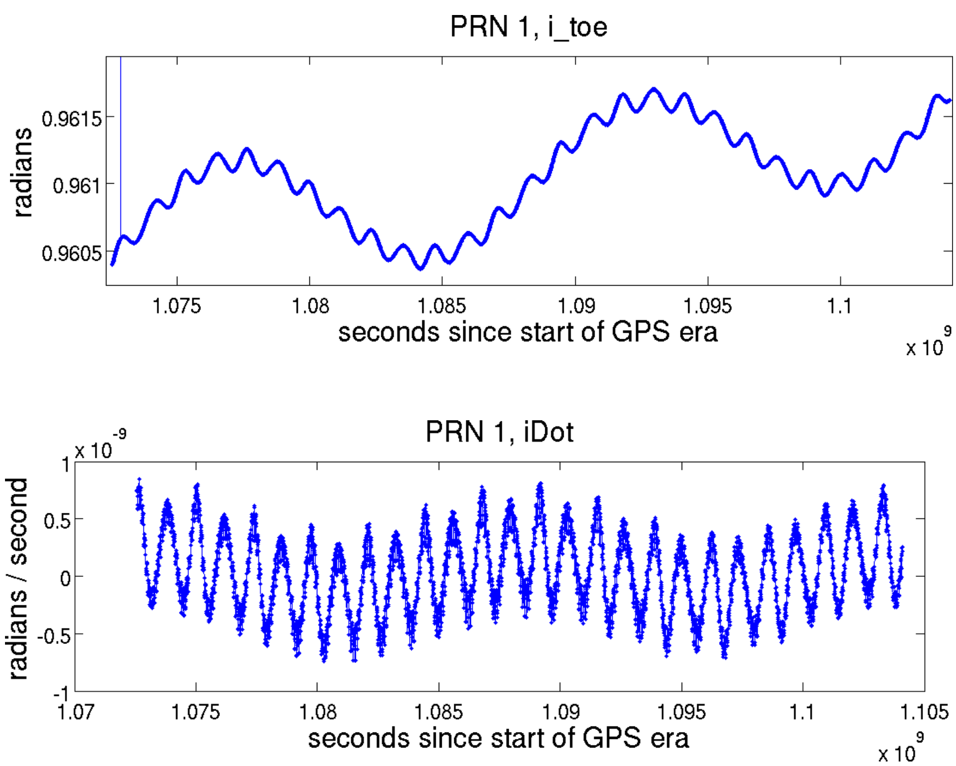


Figure 5: Variation of the orbital inclination $i(t_1)$ *at time of ephemeris, and its rate of increase* $\mathrm{d}i/\mathrm{d}t$*, over the year 2014*

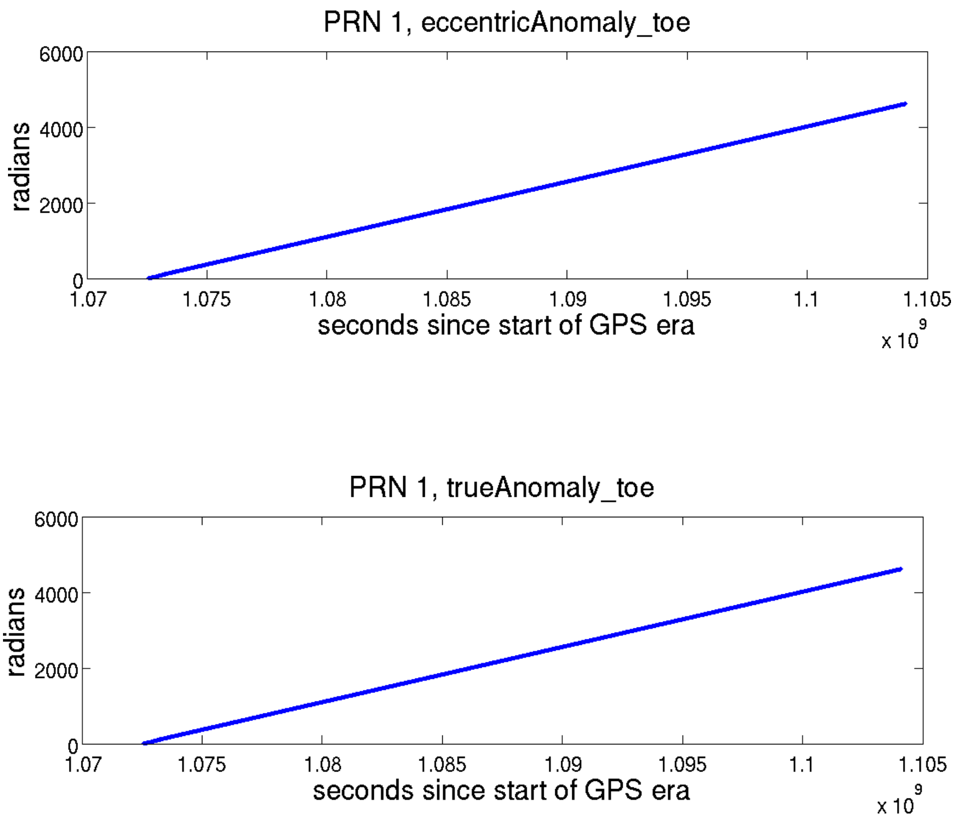


Figure 6: Variation of the eccentric and true anomalies $E(t_1)$ *and* $\theta(t_1)$ *at time of ephemeris, over the year 2014*

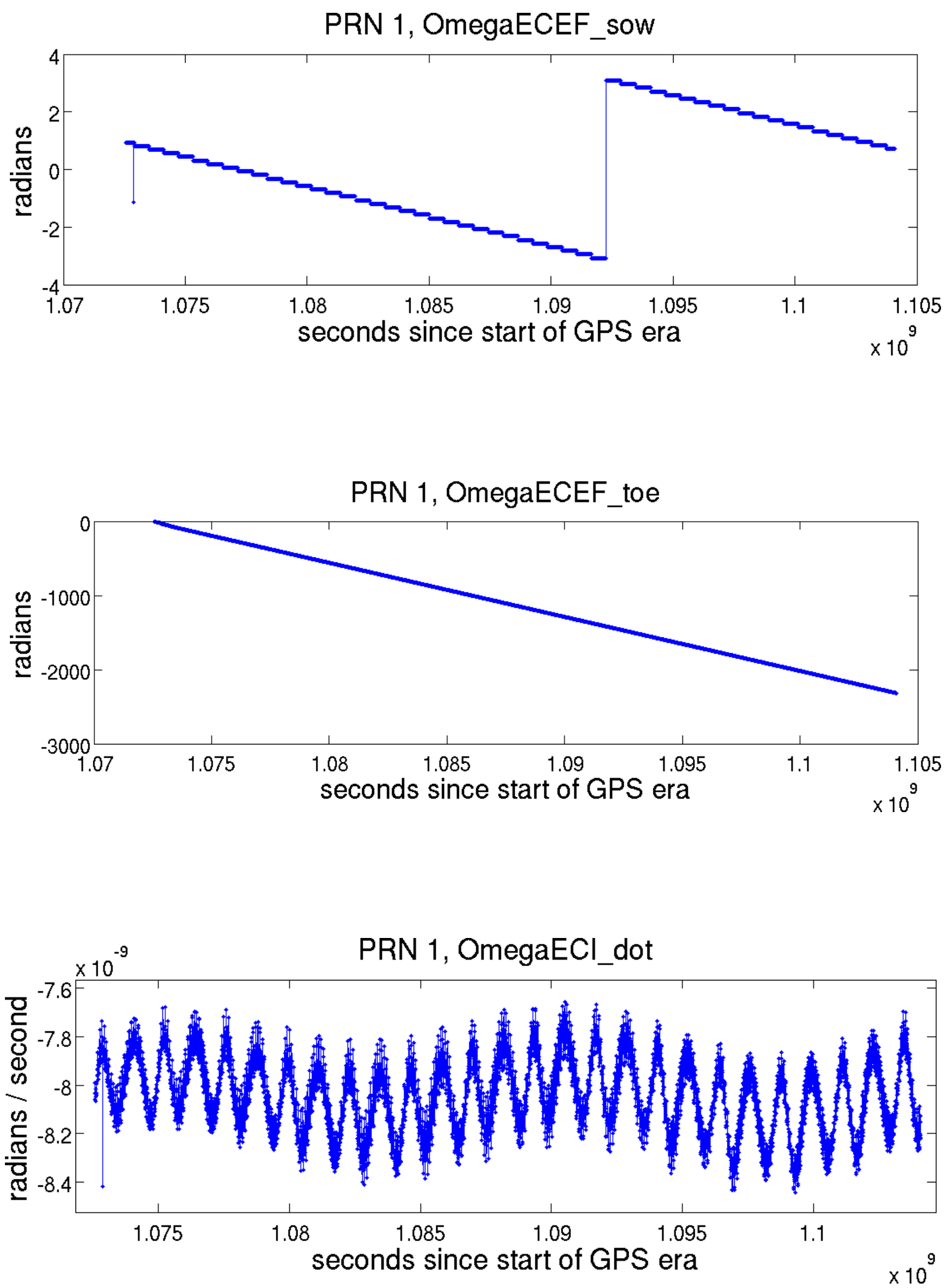


Figure 7: ***Top***: *Terrestrial longitude of the ascending node at start of week,* $\Omega_{\text{ECEF}}(t_0)$. ***Middle***: *Terrestrial longitude of the ascending node at time of ephemeris,* $\Omega_{\text{ECEF}}(t_1)$. *As pointed out on page 11, either of* $\Omega_{\text{ECEF}}(t_0)$ *(top plot) or* $\Omega_{\text{ECEF}}(t_1)$ *(middle plot) can be used to determine a satellite's position; but from the viewpoint of ephemeris extension, it's clear that the value of* $\Omega_{\text{ECEF}}(t_1)$ *is easier to predict.* ***Bottom***: *Rate of increase of the celestial longitude of the ascending node* $\dot{\Omega}_{\text{ECI}}$, *over the year 2014.*

6 Prediction using Old Ephemerides without Ephemeris Extension

A satellite's position can be predicted from any ephemeris. In the absence of orbital perturbations and satellite health issues, *any* ephemeris would be just as reliable as any other. But the ever-present perturbations force satellite orbits away from being exactly keplerian, and so the more recent the ephemeris, the more reliable the position prediction is expected to be. We must then answer the question: if a receiver powers up at a given time, say 2014-11-25-0-0-0, how accurately might it estimate its position using an old ephemeris?

We address this question by predicting the positions of three representative satellites using ephemerides of different ages, and infer how the accuracy of those positions affects the receiver's estimate of its own position. In particular, we calculate position, range, bearing, and elevation to a given GPS satellite from a specified location on Earth at 2014-11-25-0-0-0, for various choices of ephemeris date that range from current to 60 days in the past. We then plot these positions-ranges-bearings-elevations versus ephemeris age, offsetting the results to be relative to those predicted by the current (i.e. most recent) ephemeris in each case, which is taken as the ground truth. The first point of each plot is then $(0, 0)$, because this offset is zero for the most recent ephemeris, by definition. Here, by plotting a position offset is meant plotting the distance between the satellite's predicted position and its ground-truth position.

The first satellite chosen is PRN 1. The receiver position is arbitrarily chosen as latitude $-34.73037°$, longitude $138.64526°$, height above sea level 35 metres. The observation time is 2014-11-25-0-0-0. For each day in the period covering 60 days prior to 2014-11-25-0-0-0, we search the Rinex files for the ephemeris closest in time prior to UTC 0:00:00 on that day. We then use this ephemeris to predict the position-range-bearing-elevation of PRN 1 at 2014-11-25-0-0-0. What results are 61 ranges, bearings, elevations, for the 61 ephemerides that range in age from zero to 60 days. Now subtract from each of these the ground truth, meaning the value calculated from the age-zero ephemeris (i.e. that effective at 2014-11-25-0-0-0), and finally plot the resulting errors in position, range, bearing, and elevation as functions of ephemeris age. Figure 8 shows the resulting four plots. As expected, the accuracy of the predicted position degrades with ephemeris age. Most of the error in prediction is cross-range, which is evident as the position errors are typically much larger than the (down-)range errors.

Figure 9 shows analogous results for PRN 20. Figure 10 shows analogous results calculated for PRN 25, but for an observation time of 2015-2-5-0-0-0. The oscillations here are not as sinusoidal as in the previous two figures.

These plots show behaviour that appears to be typical of all satellites, and is probably not confined to the start of 2015.

Figure 11 reproduces the position and down-range plots, but this time zooming in on the first 8 days. This figure forms a basis of comparison with the ephemeris-extension work described in the next section.

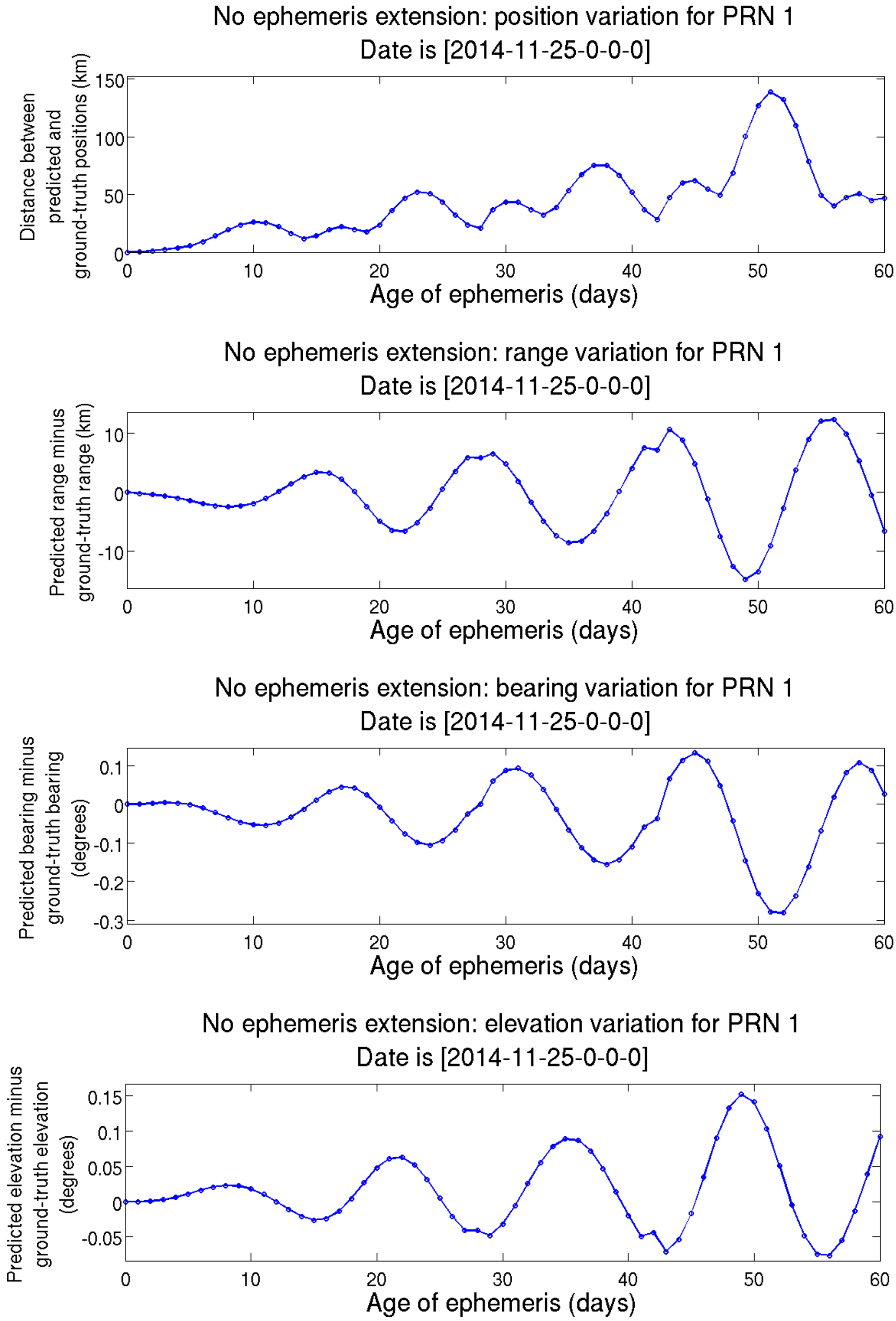


Figure 8: Position, range, bearing, and elevation offsets from ground truth for satellite PRN 1 calculated using ephemerides whose ages range from current to 60 days old. The ground truth is taken to be the value based on the most recent ephemeris. This means the plots always begin at $(0, 0)$.

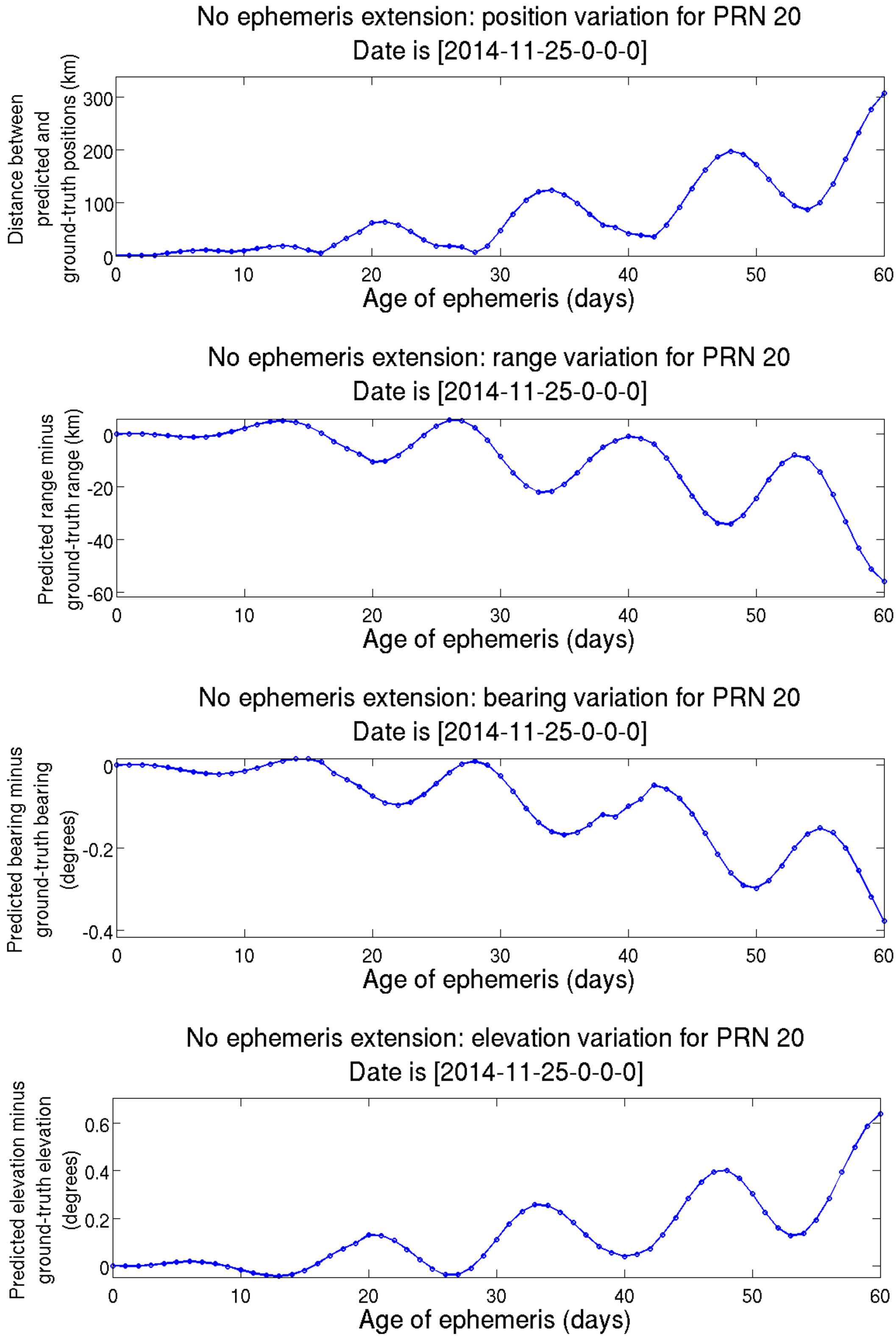


Figure 9: Position, range, bearing, and elevation offsets from ground truth for satellite PRN 20 calculated using ephemerides whose ages range from current to 60 days old. The ground truth is taken to be the value based on the most recent ephemeris. This means the plots always begin at $(0, 0)$.

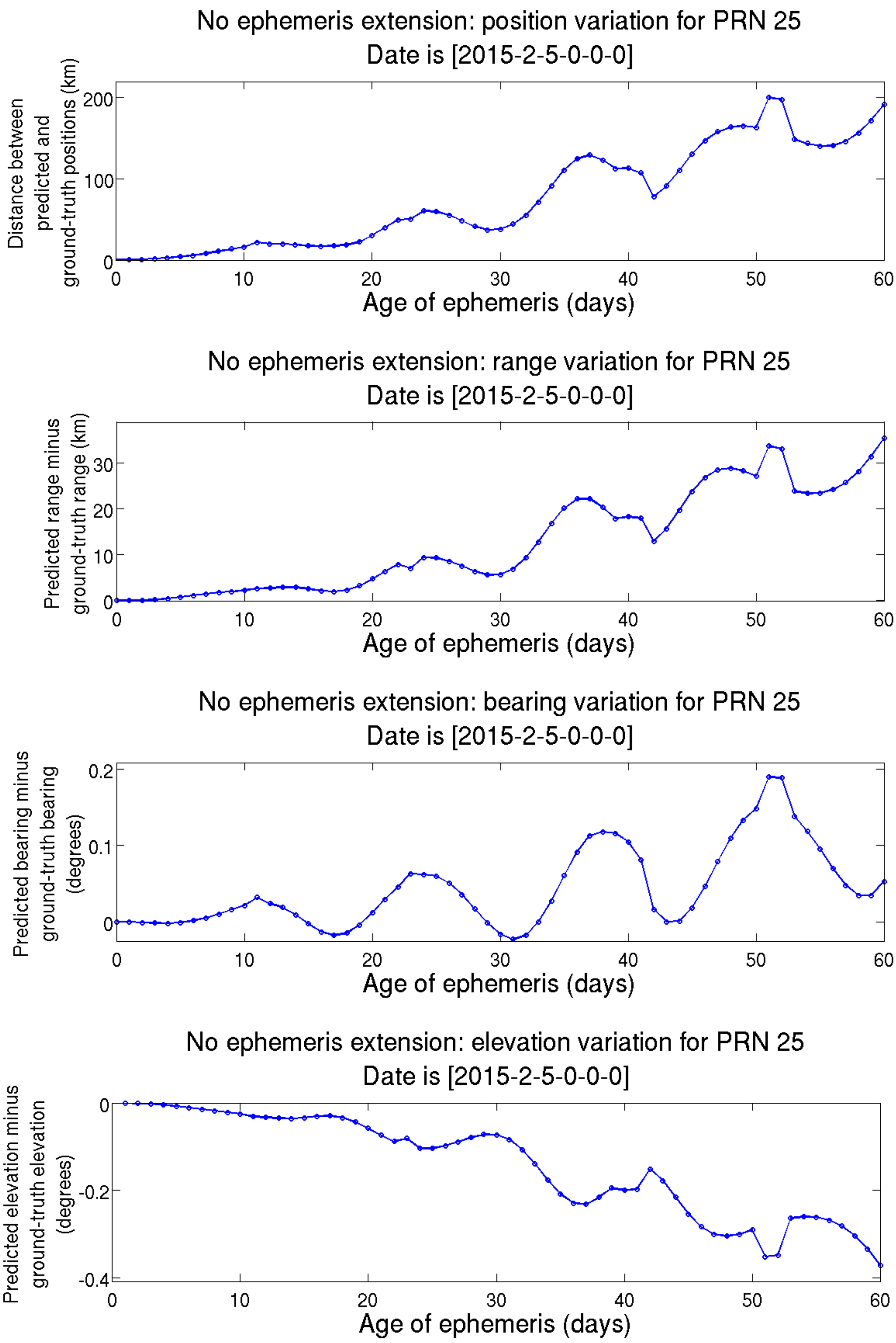


Figure 10: Position, range, bearing, and elevation offsets from ground truth for satellite PRN 25 calculated using ephemerides whose ages range from current to 60 days old. The ground truth is taken to be the value based on the most recent ephemeris. This means the plots always begin at $(0, 0)$.

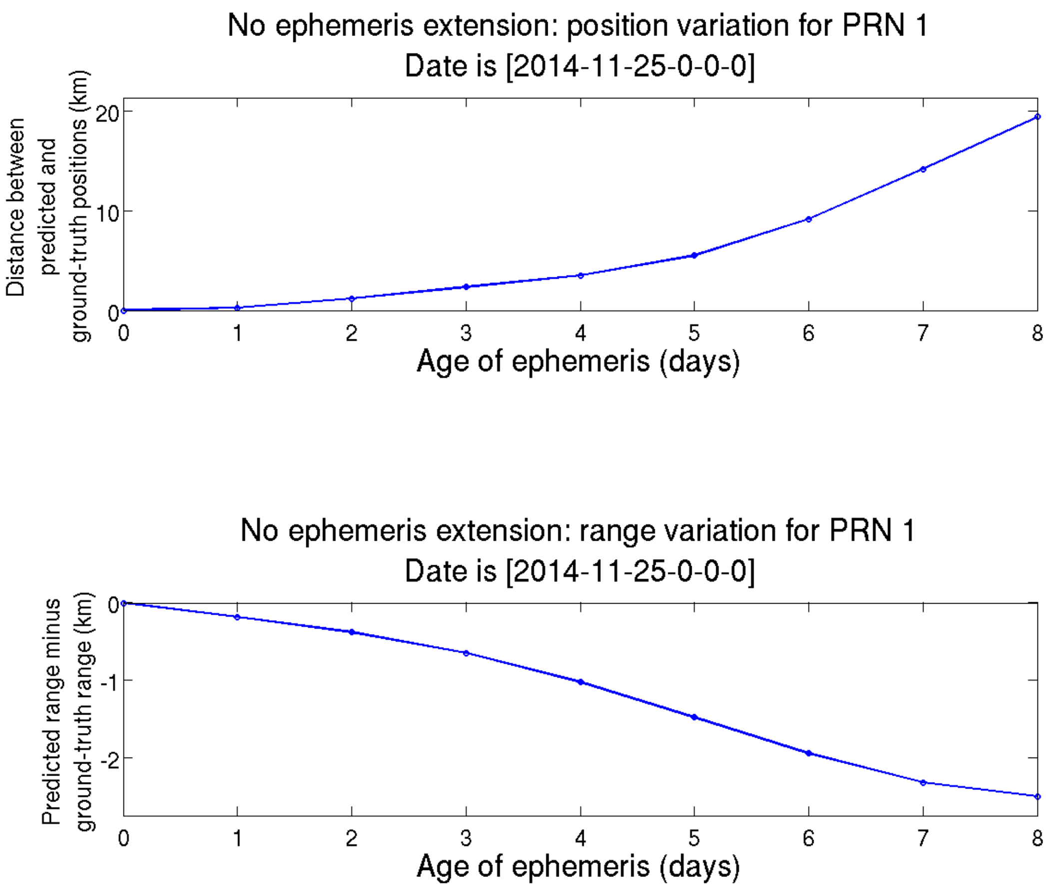


Figure 11: Position and range offsets from ground truth for satellite PRN 1 calculated using ephemerides whose ages range from current to 8 days old. (I.e. this is a zoom of Figure 8.) The ground truth is taken to be the value based on the most recent ephemeris. This means the plots always begin at $(0, 0)$.

7 Implementing an Ephemeris Extension

With a view to implementing an ephemeris extension, the receiver can extrapolate forward in time only those ephemeris parameters that absolutely must be extrapolated to ensure adequate precision is maintained. The values of the remaining parameters are acceptably insensitive to the calculation of a satellite's position, and so can be sourced from the most recent ephemeris available.

To determine the crucial parameters that must be extrapolated, we now calculate a satellite's position using an ephemeris of zero age, while perturbing each parameter in turn. The results of Section 2 show that of prime importance is to focus on how the perturbation affects the value of $r(t)$, the satellite's ("down-") range. Perturbing each parameter in isolation is a limited test of sensitivity to that parameter, in that we can infer nothing about how the calculation of a satellite's position is affected by perturbing many parameters simultaneously. But this sensitivity test is at least manageable.

The parameters $M(t_1)$ and $\Omega_{\text{ECEF}}(t_1)$ (Figures 3 and 7) vary linearly with time to a good approximation, and so their sensitivity needn't be investigated. Each of the other parameters is perturbed by adding to it one of two choices of perturbation. The first choice is "global", in that it tests a parameter's sensitivity relative to all values it can take: the perturbation is the standard deviation of all values of that parameter for the year 2014. For this choice, an arbitrary time is chosen at which to predict the satellite's position, being 2014-11-25-0-0-0; an ephemeris exists for this time, so that the time since the ephemeris, $t - t_1$ on page 6, is zero. This choice of zero elapsed time most directly allows a test of perturbations to certain of the ephemeris parameters, but it does not allow perturbing the three parameters whose influence on the satellite's position depends directly on $t - t_1$: these are ΔN, $\mathrm{d}i/\mathrm{d}t$, and $\dot{\Omega}_{\text{ECI}}$. Because a new ephemeris is issued every two hours, we can address this by choosing a second time to be one hour after the first: 2014-11-25-1-0-0, and recalculate the effect of each perturbation at this time. We then choose a third time to be almost two hours after the first: 2014-11-25-1-59-0 (just before 2:00:00 a.m. to avoid using the new ephemeris issued at this time), and again recalculate the effect of each perturbation at this time.

A second choice of perturbation is "local", in that it tests a parameter's sensitivity relative only to its more recent values. The perturbation is then set to be the standard deviation of only the most recent, say, 20 values of the parameter.

Table 2 shows the results of perturbing each parameter in turn. Columns 2–4 use the first perturbation: the standard deviation of the whole of 2014. The last column gives the results for the second perturbation. The parameters whose values are most crucial for ephemeris extension based on the "global" standard-deviation test are e, $i(t_1)$, and ω. The second column shows that perturbations to ΔN, $\mathrm{d}i/\mathrm{d}t$, and $\dot{\Omega}_{\text{ECI}}$ have no effect on the range estimate when $t - t_1 = 0$—as expected, since for such a zero value of $t - t_1$, equations (3.9), (4.3), and (3.7) (in which $\dot{\Omega}_{\text{ECI}}$ is written by the GPS specification as $\dot{\Omega}$) show that ΔN, $\mathrm{d}i/\mathrm{d}t$, and $\dot{\Omega}_{\text{ECI}}$ then indeed have no effect on the range estimate. Perturbations to these last three parameters show up in the third and fourth columns of the table. These are similar to those of column 2, and it's now clear that consideration of $\Delta N, \mathrm{d}i/\mathrm{d}t$, and $\dot{\Omega}_{\text{ECI}}$ is not important to ephemeris extension.

Note that extending by zero time (the second column of Table 2) need not reproduce the ground truth exactly, and it certainly does not in the case of e, $i(t_1)$, and ω. The reason is that extrapolating the most recent values of any ephemeris parameter involves fitting a curve that needn't pass through any of the points, so that it will almost always give an "extrapolate

Table 2: Perturbed range minus ground-truth range for satellite PRN 1 on 2014-11-25 at three times with two choices of perturbation size. Each perturbed range results from perturbing a single parameter in the first column. For columns 2 to 4, this perturbation is the standard deviation of all values of the parameter for the whole of 2014; for column 5, the perturbation is the standard deviation of only the most recent 20 values of the parameter. All values have been rounded for conciseness, and the parameters that cause large changes in the range are printed in red. The last column is printed in blue to distinguish it from the others, because it uses a different perturbation.

Parameter perturbed	Perturbed range minus ground-truth range (metres)			
	UTC 2014-11-25			
	0:00:00	1:00:00	1:59:00	1:59:00
	Whole of 2014			Most recent 20
c_{us}	−0.5	−2	0.3	0.025
c_{uc}	−3	−0.5	−0.1	−0.04
c_{rs}	−3	−26	−23	−7
c_{rc}	−44	−18	25	2
c_{is}	−0.03	−0.3	−0.1	−0.1
c_{ic}	−0.3	−0.1	0.1	0.08
$\sqrt{a}$	76	74	77	17
e	−2200	1800	5600	35
$i(t_1)$	2550	2250	1300	11
ω	110,000	38,500	−6900	−98
ΔN	0	0.8	−0.1	−0.03
$\mathrm{d}i/\mathrm{d}t$	0	6	7	2
$\dot{\Omega}_{\text{ECI}}$	0	2	5	2

by zero" value that differs from the last value in the set.

In the context of the first, "global", perturbation, consider first extrapolating many months of values, using eccentricity as an example. Assemble all values of eccentricity for some period, and perform a numerical extrapolation of these. For the period shown in Figure 3, the eccentricity follows a mildly increasing slope with a small oscillatory variation. The long-term increase must eventually end and might be followed by a long-term decrease, so we might model the eccentricity in the following way. Least-squares fit a straight line, then subtract this and fit a sinusoid to the residual. Now fit and subtract more sinusoids until the residual is reduced to the noise level.

But this way of modelling the eccentricity over a period of several months simply uses too much noisy data, giving a poor fit whose use for extrapolating the eccentricity is questionable. Although Table 2 draws attention to e, $i(t_1)$, and ω, inspection of Figures 3 to 5 shows that these parameters have a large variation over the year, so that the red values in Table 2 are actually pessimistic: it would be better to model only their most recent values. This is the reason for testing sensitivity with the above "local" perturbation that uses only the most recent 20 values of each parameter. We now see that perhaps only ω has to be extrapolated carefully. This can be done by least-squares fitting a straight line or parabola to its values.

Fitting higher-order polynomials gives disastrous results, as the polynomial tends to "blow up" immediately outside the fitting region.

As an aside, note that a discrete Fourier transform cannot be used to fit sinusoids to data for the purpose of extrapolation. Nor can the amplitude-frequency-phase triplet of parameters describing each sinusoid be reliably fitted with a least-squares approach, because a sinusoid's periodicity gives rise to multiple minima in the relevant sum of squared residuals that can cause the least-squares fit to return false values for the triplet. We can establish approximate values of amplitude-frequency-phase by trial and error, then perform a grid search of a "small" region of the 3-dimensional amplitude-frequency-phase space around these values, finally using the three values that globally minimise the sum of squared residuals. Or a Monte Carlo approach can be used, where the 3-dimensional space is randomly sampled.

7.1 Details of the Extension

Given this extrapolation of the last 20 values of e and ω, together with easy least-squares-line extrapolations of $M(t_1)$ and $\Omega_{\text{ECEF}}(t_1)$, we can begin to test an ephemeris extension in the following way. We first establish the ground truth of the ECEF position of PRN 1 at a "receiver deployment time" of 2014-11-25-0-0-0 using a current ephemeris. We then extrapolate the recent history of each parameter across some chosen period of "receiver down time" before the receiver deployment time, and use the extrapolated results to predict the ECEF position of PRN 1 at the receiver deployment time. This position can now be compared with the ground truth, which we do by calculating the distance between the extended position and ground-truth position. The same can be done for the (down-) range, producing a difference between the extension estimate and the ground-truth value.

Figure 12 shows a typical example of the extension being performed by extrapolating the last-available 20 ephemerides. The ephemeris extension is plotted in red. A good ephemeris extension is indicated by the "y values" being close to zero. As noted on the previous page in reference to Table 2, the first point of the red curve (i.e. at time zero) need not have a "y value" of zero, because extending by zero days will generally not reproduce the ground truth; a curve fitted through the most recent values of any ephemeris parameter needn't pass through any of them, and so will almost always give an "extrapolate by zero" value that differs from the last value in the set.

For comparison, the "non-extension" prediction from Figures 8 and 11 is included in blue. To reiterate, a point of each colour on each of the plots is produced in the following way. Consider the "distance" plot at the top of Figure 12, at a time of "7 days since last ephemeris". To find the value of the red point for plotting at this time, we set the last-available ephemeris to have appeared at 2014-11-18-0-0-0, being 7 days before 2014-11-25-0-0-0. We collect the 20 ephemerides previous to this last-available ephemeris, then linearly extrapolate $M(t_1)$, $\Omega_{\text{ECEF}}(t_1)$, e, and ω to the receiver deployment time, 2014-11-25-0-0-0. For the remaining parameters we simply use the values in the 2014-11-18-0-0-0 ephemeris. We now predict "zero time ahead from 2014-11-25-0-0-0" using these values. To find the value of the blue point for plotting at "7 days since last ephemeris", we simply predict 7 days ahead from the ephemeris at 2014-11-18-0-0-0.

Compare the resulting red and blue curves in Figure 12. No really large gain results from the ephemeris extension as outlined above: first, in both cases the position estimate degrades by nearly 15 km after a week. More important is the range plot. Here, PRN 1's range estimate

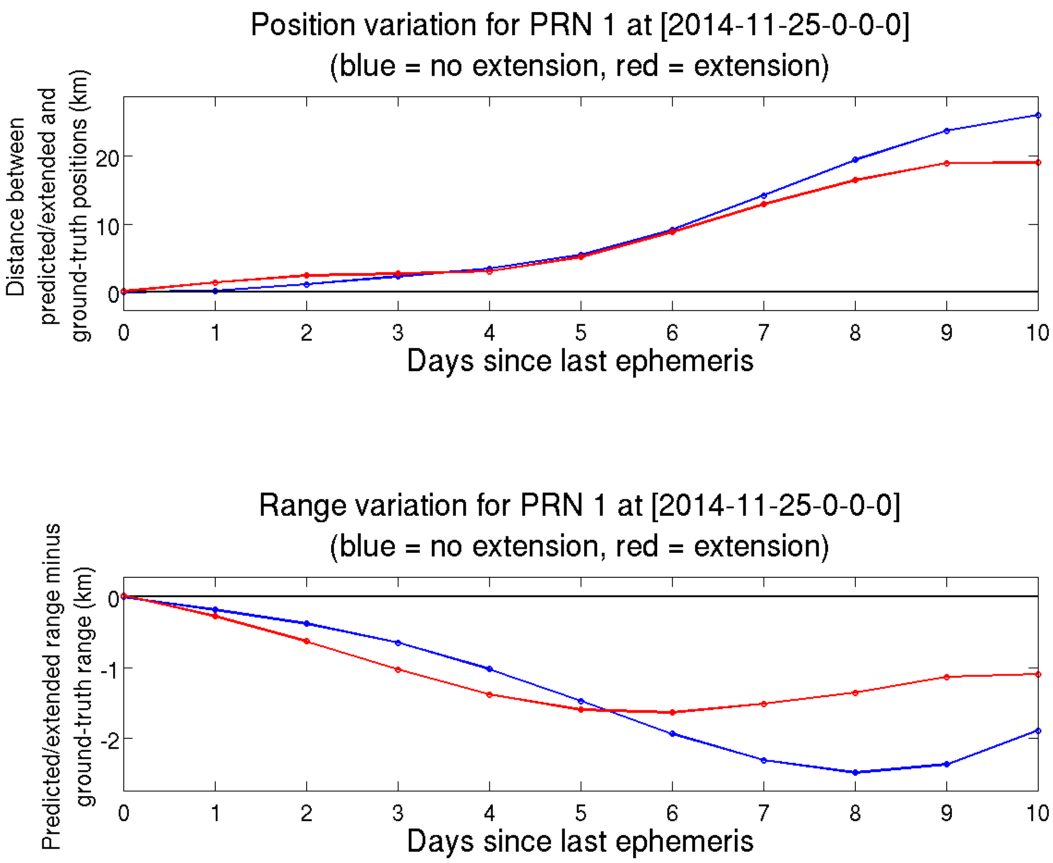


Figure 12: Position and range offsets from ground truth for satellite PRN 1 calculated using ephemeris extension (red) over receiver down times ranging from 0 to 10 days. The ground truth is taken to be the value based on the most recent ephemeris. This means the plots always begin at $(0, 0)$. *For comparison (in blue) are shown the "non-extension" predictions from Figures 8 and 11.*

degrades by about 1.5 km after a week when using extension (red), versus 2.5 km after a week when using simple predicting from the last-known ephemeris (blue). But for receiver down times of less than 5 days, the blue curve is closer to zero than the red curve: that is, the simple predicting actually does better than the ephemeris extension.

Figure 13 repeats the above analysis, but uses 10 ephemerides instead of 20 to perform the extrapolation for the extension. The results are similar to the 20-ephemerides case, presumably because we are using linear extrapolations.

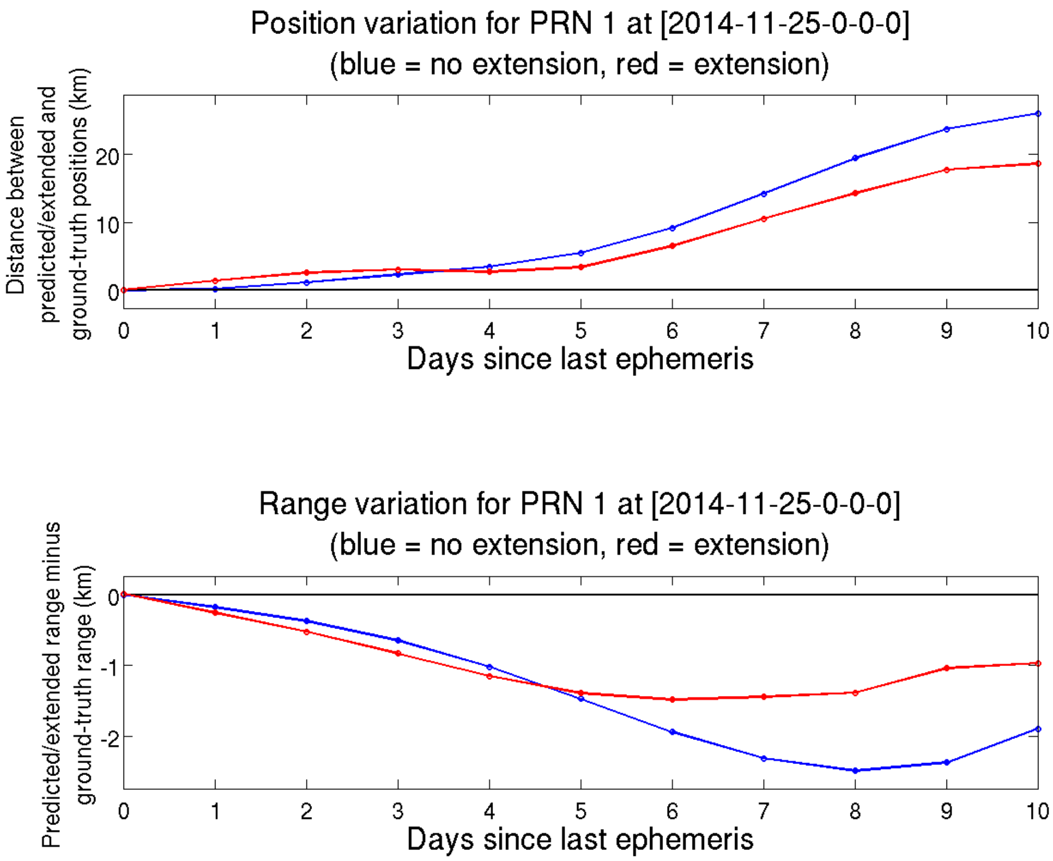


Figure 13: Rerun of the analysis that produced Figure 12, but now extrapolating from 10 ephemeris values instead of 20

7.2 Dependence of Extension Results on Extrapolation Polynomial

For the extension, $M(t_1)$ and $\Omega_{\text{ECEF}}(t_1)$ are reliably fitted with a least-squares line, whereas e and ω might be fitted with any polynomial; they have been fitted with a straight line in the previous figures. Cubic and higher-order polynomial fits tend to diverge catastrophically outside the fitting region, so we consider that fitting e and ω with only either a line or a quadratic will give any chance of a reasonable extrapolation.

Figure 14 shows range plots for PRN 1, with the four combinations of line and quadratic to e and ω. In each case, 10 ephemeris points are used for the extension. Clearly, fitting ω with a quadratic (the 2^{nd} and 4^{th} plots in the figure) gives poor results.

Figure 15 is similar to Figure 14, but now using 20 ephemeris points for the extension. The quadratic fit to ω (extension case: red curve) doesn't fail so drastically here compared to simple predicting (blue curve), but the simple predicting is still better (blue is generally closer to zero), at least for the first few days.

Figure 16 shows shows similar results for PRN 17, with 10 ephemerides.

Figure 17 shows shows similar results for PRN 17, with 20 ephemerides.

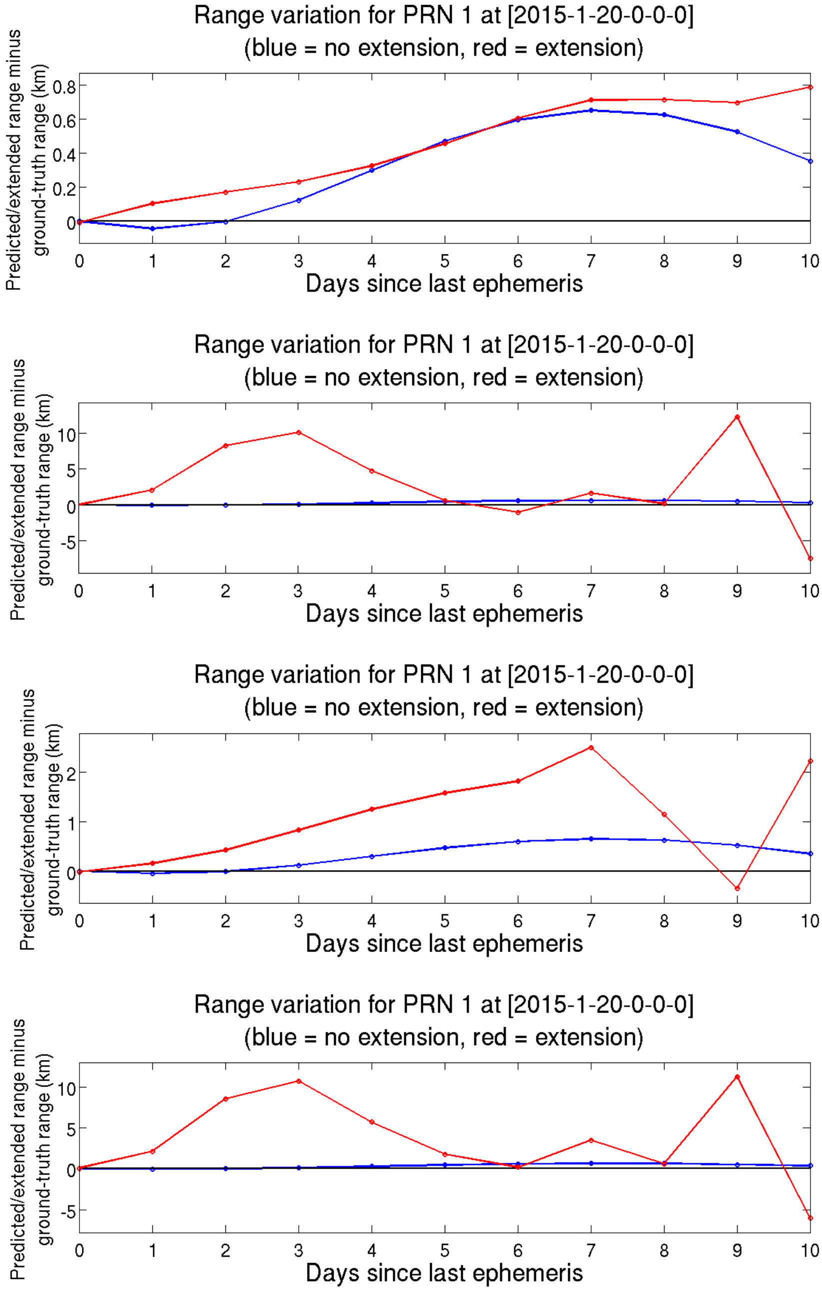


Figure 14: Range plots for PRN 1, with 10 values of e and ω being fitted respectively, and from top to bottom, with: line/line, line/quadratic, quadratic/line, and quadratic/quadratic. Note that the blue curves in all plots are the same curve; scaling makes them look different from one another.

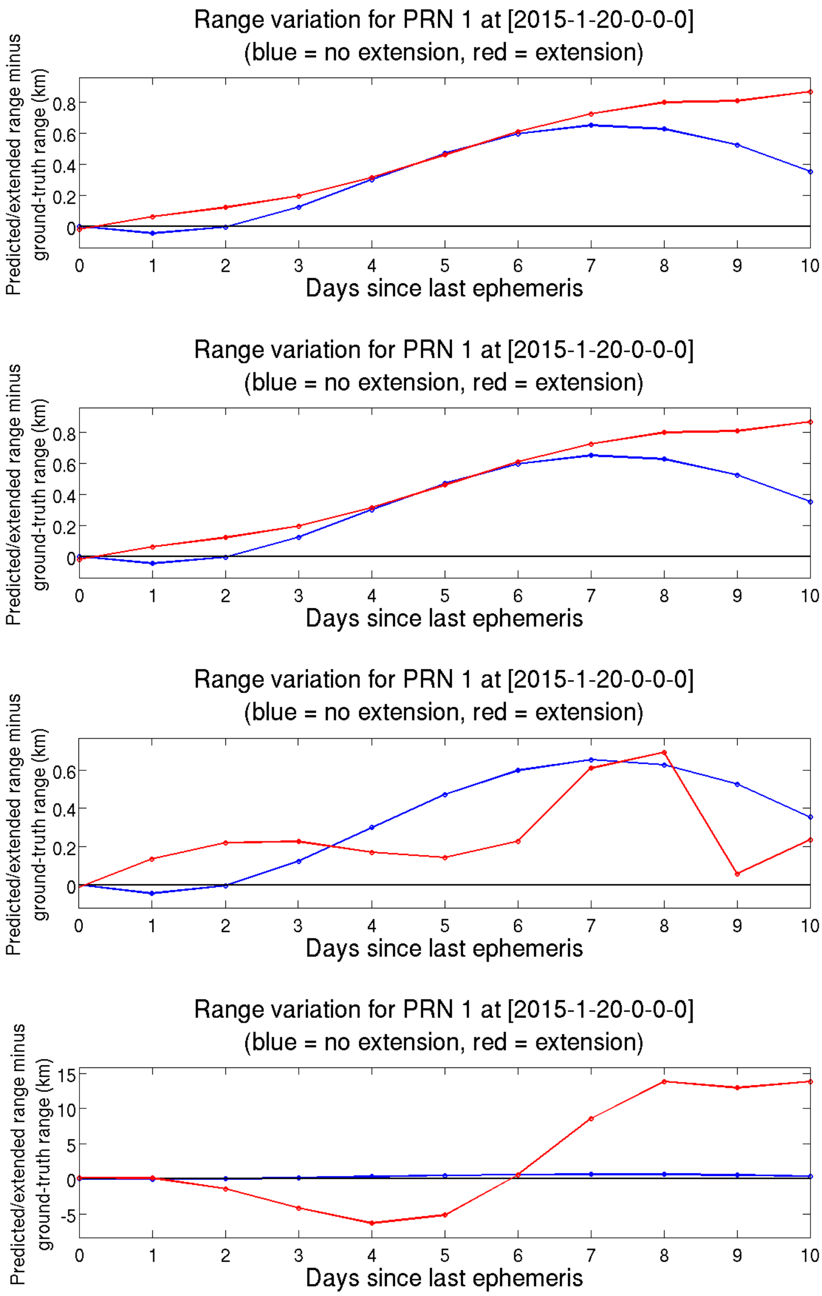


Figure 15: Range plots for PRN 1, with 20 values of e and ω being fitted respectively, and from top to bottom, with: line/line, line/quadratic, quadratic/line, and quadratic/quadratic. The blue curves differ only in their scaling.

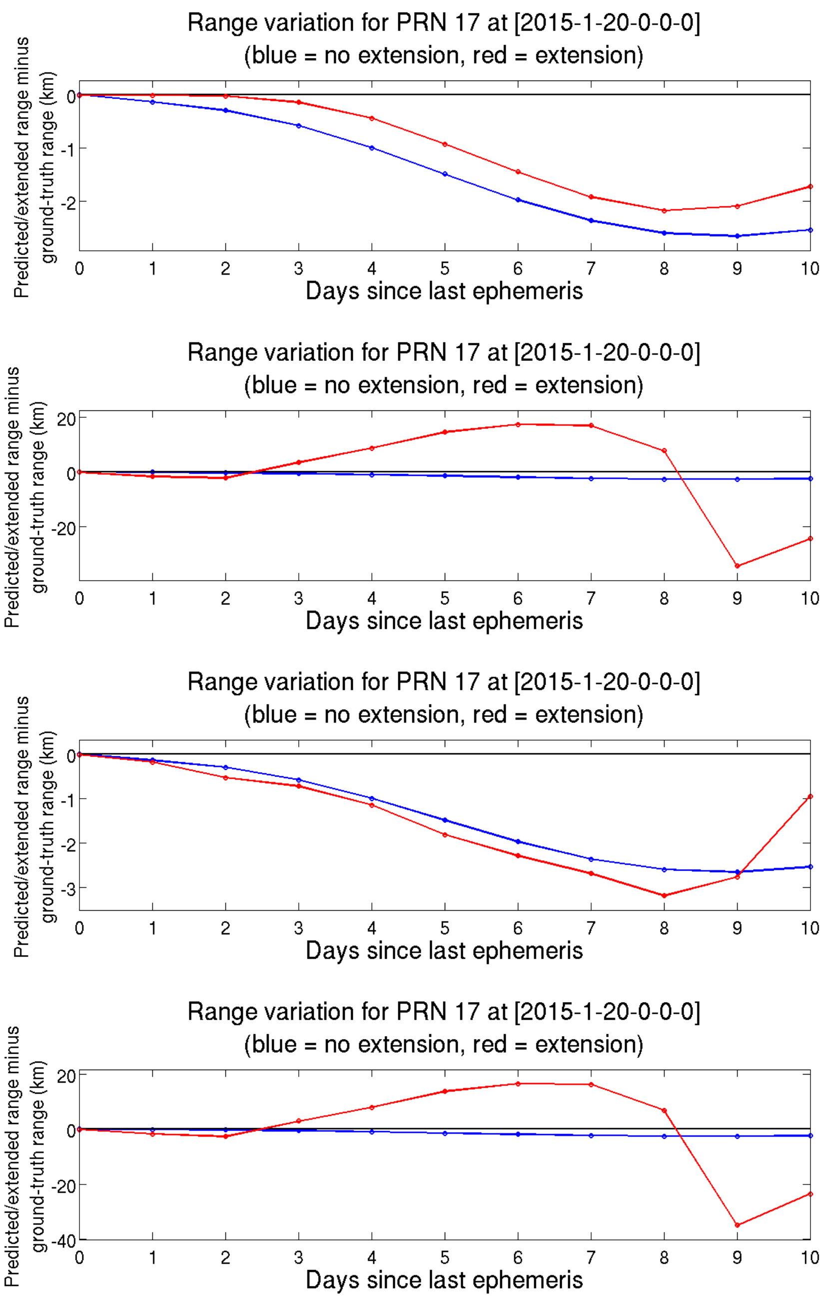


Figure 16: Range plots for PRN 17, with 10 values of e and ω being fitted respectively, and from top to bottom, with: line/line, line/quadratic, quadratic/line, and quadratic/quadratic. The blue curves differ only in their scaling.

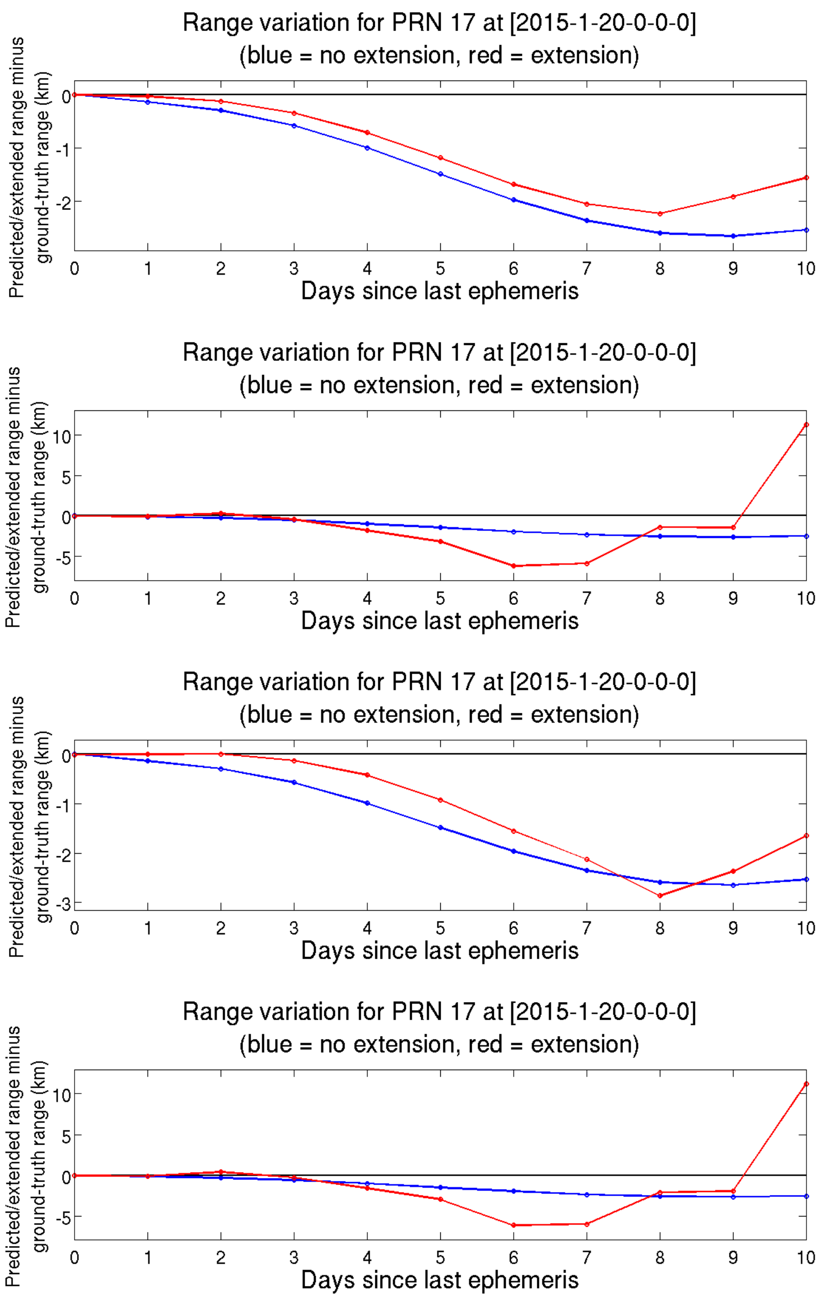


Figure 17: Range plots for PRN 17, with 20 values of e and ω being fitted respectively, and from top to bottom, with: line/line, line/quadratic, quadratic/line, and quadratic/quadratic. The blue curves differ only in their scaling.

7.3 Dependence of Results on Receiver Deployment Time for One PRN

Unfortunately, plots such as those of Figures 12 and 13 vary wildly as a function of receiver deployment time, even for one PRN. Figure 18 on the facing page shows range plots of the type shown in Figures 12 and 13, but now for a representative selection of receiver deployment times. The extension is based on linear fitting to both e and ω on 10 ephemeris points. The red extension curve sometimes gives better results than the blue simple-predicting curve (i.e. red is closer to the zero line) and sometimes worse.

8 Final Comments

A key point of Section 1.1 is that extending an old ephemeris is not guaranteed to give a better estimate of satellite position than that obtainable with no ephemeris extension. Extending the old ephemeris purely mathematically will introduce a "mathematical error" into the predictions, whereas predicting from an old ephemeris for a non-keplerian orbit (such as that of all GPS satellites) implies that we are evolving its parameters in a keplerian way, which will have the effect of using slightly incorrect *physics*, and this will again introduce errors into the prediction. This report has shown that the "mathematical error" introduced by the extension studied above is no better than the "physical error" that would occur if an old ephemeris were used with no extension, at least when the old ephemeris has a realistic age of no more than a few days.

In conclusion, I suggest that any implementation of ephemeris extension will require a more sophisticated type of extension than the purely mathematical extrapolation studied in this report. An ephemeris can presumably be extended to some useful level of accuracy if the physics of all large perturbations to the satellite orbits is modelled accurately. To this effect, the positions of Sun and Moon would be the easiest parameters to acquire. More difficult would be to continually update knowledge of solar wind pressure and Earth's tides, if indeed any knowledge of these is necessary. The cost of such a full-time collection of these parameters is something to be considered.

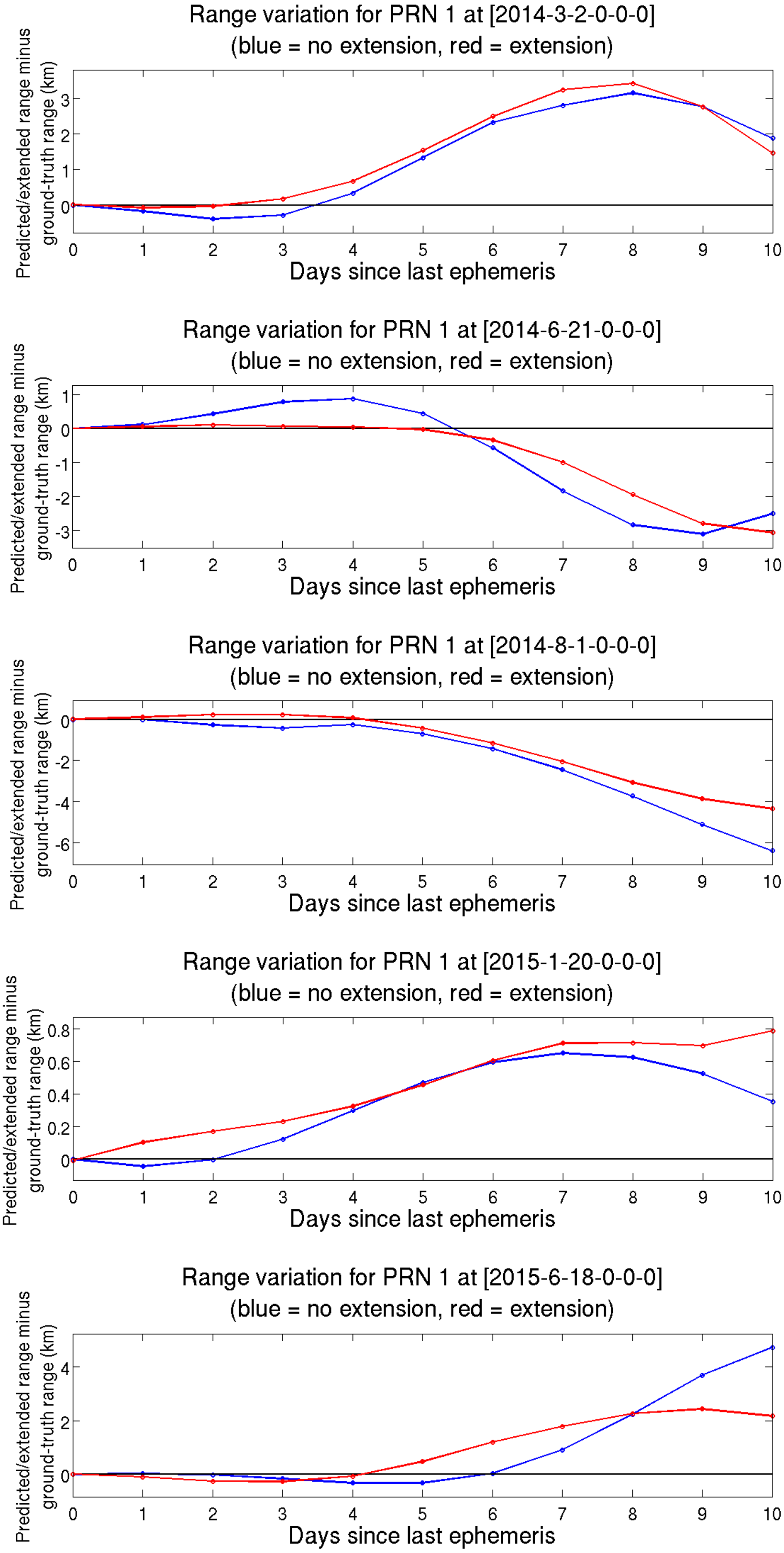


Figure 18: Range plots for PRN 1 for a representative selection of receiver deployment times show much variation across all times

9 References

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[2] F. van Diggelen (2009) *A-GPS: Assisted GPS, GNSS, and SBAS*, Artech House. See Chapter 8, *Ephemeris Extension, Long-Term Orbits*

[3] H-S. Wang (2012) GPS Ephemeris Extension Using Method of Averaging, *Recent Patents on Space Technology*, **2**, pp. 145–151.

[4] R. Prasad, H.K. Kuga (2011?) Precise predicted extended ephemeris solution for GPS, http://www2.dem.inpe.br/hkk/2011/ICNPAA-67%20Prasad.pdf. A note for physicists: this and the previous citation have the wrong sign for Earth's gravitational potential. This wrong sign has become very common in the subject's documentation, as well as in anything promulgated by the International Astronomical Union. A consequence is that tesseral harmonic coefficients, used in standard expressions for Earth's gravitational potential, appear to be defined with different signs by different authors, depending on whether they have used the correct sign for Earth's potential. I find this situation to be problematic because it makes the resulting expressions for Earth's gravity untrustworthy.

[5] Global Positioning System Directorate, Systems Engineering & Integration Interface Specification, IS-GPS-200F (21/9/2011)

[6] US Naval Observatory Circular No. 179 (October 2005), freely available from their web site. See the paragraph after their (2.10).

[7] Wells et al. (November 1999) Guide to GPS Positioning, available at http://www2.unb.ca/gge/Pubs/LN58S.pdf. See Section 5.21.

[8] See https://igscb.jpl.nasa.gov/igscb/data/format/rinex210.txt, Table A3.

[9] For discussion of this, see Don Koks (2012), A Pseudo-Reversing Theorem for Rotation and its Application to Orientation Theory. DSTO-TR-2675, Melbourne, Vic., Defence Science and Technology Organisation (Australia).

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2. TITLE

An Investigation into the Possibility of Numerical Ephemeris Extension for GPS

3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION)

Document	(U)
Title	(U)
Abstract	(U)

4. AUTHOR

Don Koks

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PO Box 1500
Edinburgh, SA 5111, Australia

6a. DST Group NUMBER	6b. AR NUMBER	6c. TYPE OF REPORT	7. DOCUMENT DATE
DST-Group–RR–0443	016–954	Research Report	September 2017

8. Objective ID	9. TASK NUMBER	10. TASK SPONSOR
	07/228	

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19. ABSTRACT

We investigate the possibility of a purely numerical extrapolation of parameters to extend the period of validity of any given set of GPS ephemeris parameters. The sought-for benefit is a faster time to first fix for a user who has been away from GPS satellite visibility for some days, and who thus holds outdated ephemeris data. Such a numerical extrapolation is not guaranteed a priori to result in more accurate satellite position forecasting, and indeed we show that for the extrapolation schemes chosen in this report, it does not result in more accurate forecasting. It therefore will not give any statistical improvement to the time taken for the above user to obtain a position fix.