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A Study of Relativistic Bounds on Clock Synchronisation on Earth

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ABSTRACT

We investigate the extent to which a high-precision clock might be able to synchronise another to its own time, when they are part of a network that requires time-stamping events to a very high precision. Synchronisation at an ultra-fine level is strongly subject to the rules of relativity: clocks that are at rest in a single inertial frame can (in principle) synchronise each other to any accuracy required, whereas clocks at rest on Earth or on satellites are not inertial, and hence cannot necessarily synchronise each other to an arbitrary level of accuracy. Aside from the standard result that clocks over a wide area of Earth cannot agree on the timing of events on Earth to better than some tens of nanoseconds (a result which does not contradict the success of satellite-positioning technology), we discuss simultaneity in detail, and prove a related and new result for the extent to which two clocks at rest on Earth at the same height might agree on the meaning of "now". Although the answer requires no change in current technology, it must be understood in context. This report describes that context in detail.

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Executive Summary

The question of the extent and meaning of clock synchronisation is becoming pertinent as clocks become ever more accurate, and are used in networks that place increasing demands on the accuracy of their time-stamps. At such high levels of accuracy, relativity has an important role in timing analyses. Because both gravity and Earth's rotation affect a clock's tick rate relativistically, the extent to which one clock on Earth can synchronise with another, within the bounds imposed by relativity, must be addressed. The necessary analysis will not be straightforward and will require some caveats, because synchronisation can be problematic in relativity. This scenario differs from the well-understood synchronisation of clocks by, say, a GPS satellite, where the satellite is acting as a master clock. In this report, we are assuming that no GPS satellite is available to perform the synchronisation. We must be very careful to specify the frame in which the synchronisation is being performed.

Simultaneity is an old subject in relativity, but that does not mean that all questions pertaining to it have long ago been answered. It is well understood for inertial frames, but it is less straightforward for the non-inertial frames that are relevant to a rotating Earth. All analysis in this report uses orthodox relativity: that simultaneity is always defined in an inertial frame, and if no such frame is available, then the next best thing is used if possible: a series of "momentarily comoving inertial frames".

In this report, the use of momentarily comoving inertial frames allows us to place a bound of about 10^{-19} seconds on the extent to which two clocks at rest on Earth can agree (even in principle) on the meaning of a "shared now". This time interval is far smaller than current accuracies preservable within communications networks—meaning, things are okay for now: current technology has not advanced to the level where the demands of relativity become essential to it. Nonetheless, we must not infer that simultaneity can be defined over the whole of Earth to this level. In particular, clocks that are stationed over a wide area of Earth cannot agree on the timing of events on Earth to better than some tens of nanoseconds. This report discusses how these numbers are calculated, what they mean, and why they do not clash with the success of satellite-positioning technology.

This report contains several appendices that cover well-known concepts in relativistic timing that I think are not fully explained in the literature, and are more or less absent from relativity textbooks, since these books tend to avoid discussions of precise timing in a technological context.

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Don Koks completed a doctorate in mathematical physics at Adelaide University in 1996, with a thesis describing the use of quantum statistical methods to analyse decoherence, entropy and thermal radiance in both the early universe and black hole theory. He holds a Bachelor of Science from the University of Auckland in pure and applied mathematics and physics, and a Master of Science in physics from the same university with a thesis in applied accelerator physics (proton-induced X ray and γ *ray emission for trace element analysis). He has worked on the accelerator mass spectrometry programme at the Australian National University in Canberra, and in commercial Internet development.*

Currently he is a Research Scientist with the Positioning Navigation and Timing Technologies and Systems Group in the Cyber and Electronic Warfare Division at the Defence Science and Technology Group, specialising in the relativity aspects of precision timing, satellite orbital prediction, 6-degree-of-freedom orientation concepts, and radar signal processing. He is the author of the books Explorations in Mathematical Physics *and* Microstates, Entropy and Quanta*, published by Springer.*

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1. Introduction

Many agencies around the world—both Defence and civilian—have an increasing need for precise timing. A well-known civilian example of this need is the time-tagging of monetary transactions, especially when commodities are being bought and sold in real time, meaning that money is constantly being exchanged and the precise order of transactions is of crucial importance. These transactions might occur on a global scale. Defence operations tend to be less global, such as when several entities observe a target, and must time-tag their observations if these are to be fused.

A network of high-precision clocks will tend to work best if the clocks are synchronised. Loosely, this means that they all display the same reading at the same time. Even if they do not all display the *same* reading, the synchronisation really refers to the fact that given one reading, we know the readings of all the others at that moment. We will assume that any of these known differences between readings have been removed. So, we will understand that "being synchronised" means that the clocks all display the *same reading at the same time.* We will also assume that the clocks of our network are fixed relative to each other. If a clock moves, a difference between its reading and that of other clocks (due to relativity) will be created, and this difference is generally difficult to accommodate.

Synchronising clocks at a coarse level of, say, microseconds, is a well-understood task of swapping handshake signals; but at a level of high precision (nanoseconds and beyond), the notion of "the same time" must be understood carefully, and this requires concepts of relativity. This notion of deciding or defining which events are simultaneous for entities in arbitrary motion and in a gravity field is very subtle, and in fact is not fully agreed upon by physicists. It has, for example, led to a century of debate over how relativity in rotating systems should be formulated [1], even in the absence of gravity. It makes discussions of ultra-high-precision timing difficult, and it places a limit on what synchronisation can be achieved that physicists will generally agree with, even with otherwise perfect clocks.

Speaking purely of current technology and not relativity, clocks that are synchronised will gradually drift apart, and so must be re-synchronised at various maintenance intervals. Clocks that are sited close together might be synchronised with a direct link. In both civilian and Defence applications, widely separated clocks tend to be synchronised using a set of satellites such as the Global Positioning System (GPS); but to appreciate the details of this synchronisation, we must know *which frame* the clocks are being synchronised in, of which more will be said later. A GPS receiver co-located with a to-be-synchronised clock can calculate an accurate time at the location of that clock, and this time is then used to reset the clock's display. But to retain an ability to synchronise in the event of GPS being unavailable, users of a network of widely separated clocks need another way to synchronise, perhaps via some more direct link that doesn't rely on GPS.

Regardless of their separation, then, we require an analysis of clocks that are synchronised via a direct link between them (and not connecting them to, say, a GPS satellite), to see how this relates to a synchronisation carried out using GPS. We must take care with the meaning of "time" when discussing these two types of synchronisation, because they can relate to different frames.

The undisputed fact that a disagreement *does* exist among physicists regarding the details of simultaneity is discussed in [1] and [2]. In the current report, I have established some standard notation and jargon of special relativity in Appendix A, along with a discussion of why gravity plays a role in the flow of time. In the body of the report, I discuss reference frames and difficulties of synchronisation, then calculate the extent to which two clocks on Earth can define a "shared now". I discuss details of synchronising Earth-fixed clocks and how this relates to the Sagnac effect, and then extend the analysis to include gravity. Including gravity for a real Earth makes everything less well defined, and so although some discussion of gravity is included, the special relativistic bounds established here can be treated as optimistic. Further relevant topics are included as appendices. Although these topics are mostly well known, they are more or less absent from relativity textbooks (since these seldom discuss timing), and are difficult to find even in precise-timing literature.

2. Simultaneity and the Definition of a Frame

We are given two "ideal clocks", meaning ones that keep time perfectly. Then, within the bounds of relativity, how closely can one synchronise with the other? Aside from any relevant physics, a preliminary answer is: we might aim to synchronise them only to the level of our understanding or agreement on the concept of simultaneity. Simultaneity is well defined for inertial frames, but its meaning for non-inertial motion has not found universal agreement. I take simultaneity to be a fundamental feature in special relativity, originally described by Einstein; its definition accords with physical intuition, and is a fixed concept that is not open to being redefined arbitrarily.

Probably the most important argument in favour of this definition of simultaneity as originally put forward by Einstein (and used throughout this report), is that it is validated *experimentally*. Einstein's definition of simultaneity can be inserted into the mathematics of the uniformly accelerated frame (see Section 2.2), and then into discussions of gravity proper, via Einstein's Equivalence Principle. When this is done, the predictions of Einstein's simultaneity for the readings on a set of clocks turn out to match the predictions of weak-field general relativity in a small region of space; and these predictions are verified by modern measurements. Full details are given in [3].

In contrast, many precise-timing practitioners, and some physicists, believe that simultaneity can be redefined in whatever way one chooses [4]. Such re-definitions can probably never dovetail with general relativity, in which case they are worthless. I think that the belief that simultaneity is arbitrary renders it meaningless, and also that such arbitrariness contradicts the tenets of relativity at a most basic and obvious level. If we allow an arbitrary definition even within the simplicity of an inertial frame, then we essentially create a set of meaningless and mutually contradictory statements that are of no use to anyone.

Why is the belief so widespread that simultaneity is arbitrary, given my argument above about its experimental validation? I think the reason is that my argument rests on a knowledge of the uniformly accelerated frame. This frame is discussed using arcane mathematics in dusty corners of a few books on relativity. But any real discussion of its full worth—as a flat-spacetime approximation to the frame of an everyday laboratory—is, surprisingly, essentially absent from relativity courses and textbooks.

Finally, despite the best efforts of textbook authors, a belief can still be found on the fringes of physics that two events are deemed to be simultaneous by a single observer if they are merely *seen* at the same time, irrespective of how far away each of them is. This belief is in disagreement with the most basic physics, and should have no place in journal papers; and yet, it was put forward as recently as 2017 [5, 6].

2.1. Reference Frames

Simultaneity is bound to the definition of a reference frame. A *reference frame*, or simply "frame", is a collection of observers who each record the positions and times of objects and events, each only in their close vicinity, with the following two conditions satisfied:

1. Each observer measures all other observers to be at fixed locations relative to each other. Their fixed separation defines their common ruler.

2. All observers agree on simultaneity. That is, if two events are simultaneous for one observer, then they are simultaneous for all observers. This defines their common clock.

A frame is usefully pictured as a lattice that is populated by a continuum of observers, each occupying a fixed point on the lattice and holding their own clock, who each record the positions and times of events only in their "very close" vicinity. These observers can be envisaged as continuously sending their data to a master observer, who continuously collates this information to form a global picture of all events in spacetime. It's normal to consider "observer" and "clock" to be synonymous, and so we will use "clock" when it simplifies the language in the descriptions that follow.

Non-relativistic frames obey the above two conditions only up to some approximation. Consider the two most widely used arenas in which discussions of precise timing are placed. First is the "Earth-Centred Inertial" frame (ECI), which is effectively the frame of the distant stars, in which Earth turns once per sidereal day. The ECI is, to a high degree of precision, a globally inertial frame. Second is the "Earth-Centred Earth-Fixed" frame (ECEF), which is the "everyday world" in which Earth does not turn. We'll see shortly that the ECEF fails the second condition above to a small extent, relativistically speaking, in that its inhabitants cannot agree on simultaneity to high accuracy. But, for brevity, we still call it a frame.[1]

When synchronising clocks in a given frame, we must accept that they will then not be synchronised in other frames. Whether this affects any practical precise timing will be discussed in this report. Being different frames, the ECI and ECEF have different standards of simultaneity.

Relativity's definition of simultaneity accords with our intuition: it says that two events are simultaneous if and only if two signals of equal speeds that were sent from halfway between the sites of those events will intercept the events. This idea of equal speeds immediately calls in a discussion of light travelling in an inertial frame. The subject of whether light's speed is independent of direction in an inertial frame is an old one, but we will take an inertial frame to admit no privileged directions, and so will assume that light's speed *is* independent of its direction of travel in such a frame.

Special relativity's postulated invariance of all inertial frames dictates that we must be able to synchronise a distant clock with our own by sending it a signal that sets its display to be half of the return-trip time of the signal. This synchronisation is shown in an inertial frame in Figure 1. This figure shows the world lines of two clocks, at rest relative to each other. These are ideal clocks that are considered to tick identically in the inertial frame, since this frame has no privileged position at which time might run faster than at some other position. The blue clock's time is called t_{blue}, and the red clock's time is called t_{red}. Blue sends out a light signal at $t_{\text{blue}} = 0$, and it knows from experience that a light signal bounced from Red

[1]It can, of course, certainly be treated as a frame in non-relativistic analyses such as flying aircraft over Earth's surface, or calculating the footprint and observed motion of orbiting satellites.

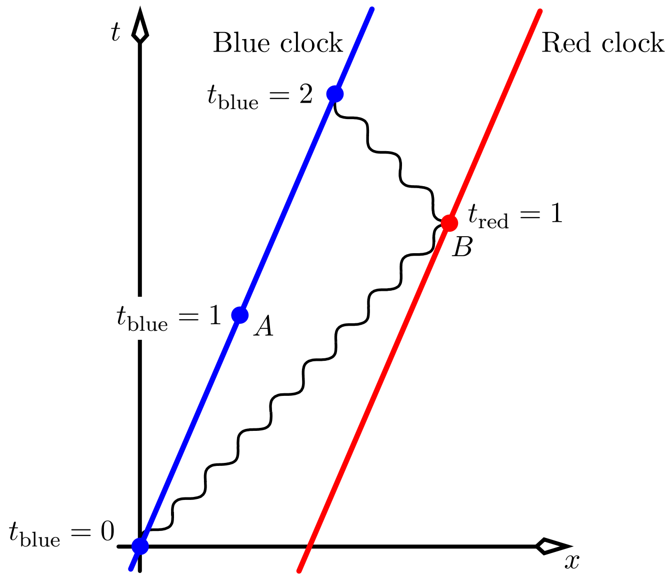

Figure 1: The procedure of synchronising two clocks that have constant and equal velocities in an inertial frame

takes 2 seconds for the return trip. In that case, the signal sent by Blue is an instruction to set Red to display $t_{\text{red}} = 1$. Events A and B are deemed to be simultaneous by Blue, because Blue says that the outbound and inbound light signals travelled equal distances with equal speeds. A similar analysis will show that in this case where the clocks have equal velocities, A and B are also deemed to be simultaneous by Red. Red and Blue thus agree on simultaneity.

The geometry of the world lines and axes in Figure 1 makes it immediately clear that in the inertial frame plotted in the figure, the two events A and B occur at *different* times. In fact, if we fill the space with a continuum of clocks that all share the same velocity (as is usually done in special relativity), then it's easy to see that all events on a line joining A and B (in both directions out to arbitrary distances) will be deemed by all the clocks to be simultaneous with A and B. This line is the common "line of simultaneity" through A and through B: the set of all events that are simultaneous with either of A and B, for the case of one spatial dimension in Figure 1. If we augment Figure 1 with another space dimension (a y axis normal to the page), all events simultaneous with A and B will lie on a "plane of simultaneity", whose normal lies in the plane of the page.

A set of clocks with equal velocities in an inertial frame will all agree on which events are simultaneous and which are not, and this property, along with the fact that they each measure the others to be at a fixed displacement from themselves, allows them to be considered as a frame. Probably only one other frame in special relativity exists for which all clocks agree on the simultaneity of events: the "uniformly accelerated frame", discussed in Section 2.2.

The procedure of synchronising clocks with constant and equal velocities in Figure 1 is sometimes referred to as "Einstein synchronisation". I think the term is misleading, because it can imply that synchronisation is an arbitrary procedure that can be redefined to suit our tastes or to get us out of a perceived bind, which is precisely what is done in [7]. Rather, synchronisation is a natural by-product of a deep and fundamental tenet of all of physics: that all inertial frames share an equal footing. It is imposed on us by physics. It is not an arbitrary choice that Einstein made, but instead is a completely natural and intuitive way that he made use of the observer-independent speed of light to draw meaning from a set of clocks.

Figure 1 is a basic tool in developing the main ideas of special relativity. In Appendix A, I provide a primer on the Lorentz transform to establish the notation and jargon used in this report. In that appendix, I establish through conventional arguments the four key ingredients of special relativity. These are time dilation, length contraction, different levels of synchronisation, and describing sets of simultaneous events (such as the line of simultaneity in Figure 1). In particular, in this report I am interested in different levels of synchronisation, and describing sets of simultaneous events. A key result from Appendix A is that in one space dimension, an inertial observer with velocity v draws parallel lines of simultaneity with slope v through all events; that is, the line joining A and B in Figure 1 has slope v. In two space dimensions, this line becomes a plane, where any (t, x, y) on this plane has its (t, x) elements on this line, with y arbitrary.

It's important to distinguish between a frame and a set of coordinates, since doing so is the reason why the Lorentz transform exists at all, as opposed to the usual Galilei transform that relates inertial frames in the non-relativistic limit. Although we can always write a Galilei transform in a relativistic context, the coordinates that it produces will not behave in the way that we expect and require good coordinates to behave. In particular, two events that are simultaneous (such as A and B in Figure 1) will not necessarily have the same Galilei time coordinate; and two events with the same Galilei time coordinate will not necessarily be simultaneous. This makes the Galilei time coordinate generally useless to describe a set of events; for example, ordering the events in a discussion about causality will be difficult when this coordinate is used. This requirement that a good set of coordinates must obey is not well know in the field of precision timing, whose practitioners tend to insist that because relativity can be expressed in tensor language (which is independent of the choice of coordinates), then any choice of coordinates is as good as any other choice. For example, reference [8] makes no distinction between arbitrary coordinate choices (simple one-to-one maps of numbers, devoid of physical content) and the real physics of relativity, which is built on establishing simultaneity and distinguishing real frames from trivial coordinate choices. It maintains that coordinate choices are sufficient in the subject. But this is akin to saying that a Galilei transform is sufficient in modern physics, with the Lorentz transform being just a distraction: clearly, a wrong statement. Tensors are certainly useful for writing equations in a form that doesn't single out a particular choice of coordinates, but this does not imply that any choice of coordinates is as physically meaningful as any other. This is proved by the above discussion: the coordinates that result from a Galilei transform are coordinates for sure, but they do not have the physical meaning that Lorentz coordinates have.

2.2. Some Observer Sets and their Simultaneity Standards

Below is a short list of observer sets in the absence of gravity, whose motions follow an ever-increasing degree of complexity. For each set, we discuss their standards of simultaneity and hence whether they form a frame. Proofs of the various statements are in principle straightforward, involving lines of simultaneity (slope v) or planes of simultaneity drawn through various events; some of this can be done qualitatively by eye. Quantitative proofs for the uniformly accelerated frame are omitted because their details can be somewhat involved, but these can be found elsewhere [9, 3].

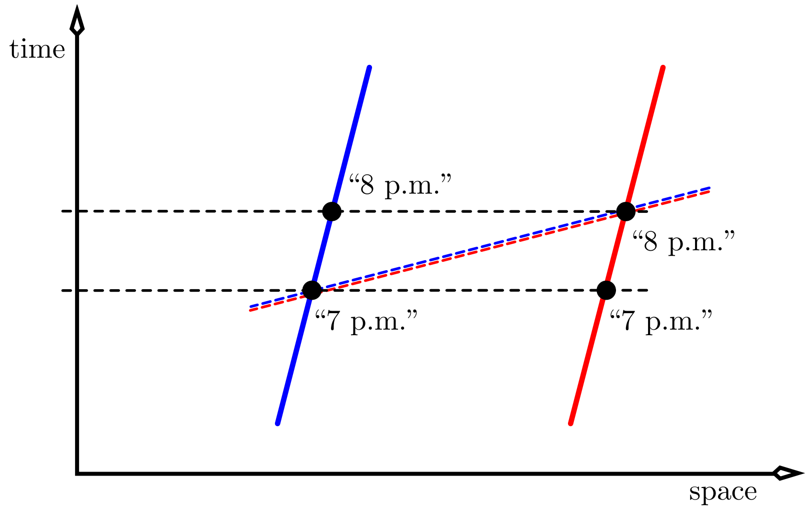

Figure 2: The blue and red lines are the world lines of a blue clock and a red clock in one space dimension. These clocks are moving in the inertial frame of the picture at the same constant velocity, and are synchronised in that frame. At the lower-left event marked with a small disk, Blue displays 7 p.m. At this event, Blue's line of simultaneity (the set of events it regards as simultaneous, according to the rules of relativity) is drawn as a blue dashed line. Blue says "When I display 7 p.m., Red displays 8 p.m." Red's line of simultaneity (red dashed) at Red's 8 p.m. is identical to Blue's line of simultaneity, and so Red says "When I display 8 p.m., Blue displays 7 p.m." Blue and Red thus share a common standard (line) of simultaneity, and can be shown each to measure the other to be at a fixed distance. Hence they define a frame. Because they do, they are at liberty to set Red's display back by one hour, so that all simultaneous events are given the same time coordinate by them.

Inertial Observers: All observers who share a common constant velocity will agree on which events are simultaneous, and will measure each other as having a fixed relative position. An example of simultaneity is shown in Figure 2, which shows the motion of two observers in an inertial frame. Observers "Blue" and "Red" agree on the simultaneity of all events. They can synchronise their clocks by having Red set his clock back one hour. Then, each can say "When my clock displays 7 p.m., the other clock also displays 7 p.m." In the inertial frame of Figure 2, Blue's clock will then display more than Red's clock, by precisely the amount vL'/c^2 of (A.6). The ability of these observers to agree on simultaneity, combined with the fact that they each measure the other to be at a fixed distance, means that they inhabit their own frame, which is of course also inertial in this case.

Identically Accelerated Observers: The next level of complexity beyond the inertial observers of Figure 2 involves two identically accelerated observers. Do they agree on the simultaneity of events? The basic idea of simultaneity in relativity are based on inertial observers. Relativity makes headway with non-inertial observers by postulating that local measurements made by a non-inertial observer are always identical to measurements of the same events that are made in his "momentarily co-moving inertial frame" (MCIF), which is the inertial frame that for a brief moment is at rest relative to him: the frame of an inertial observer whose world line is tangential to the accelerated observer's

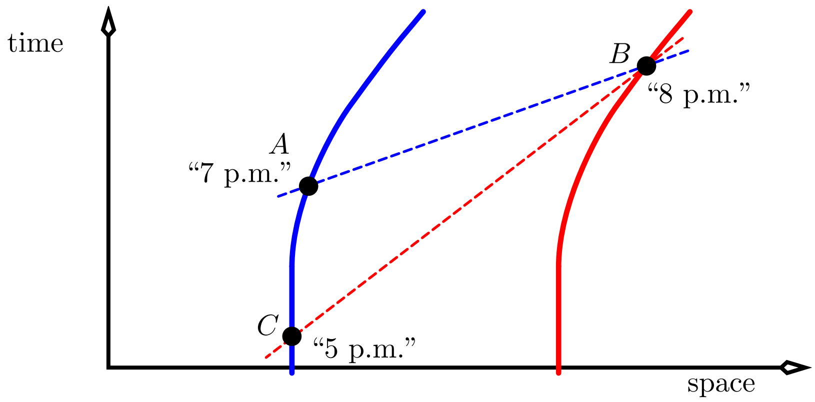

Figure 3: At event A, Blue displays 7 p.m. Blue says "When I display 7 p.m., Red displays 8 p.m." But Red says "When I display 8 p.m., Blue displays 5 p.m." These observers do not share a common standard (line) of simultaneity (they don't have a "shared now")—and also, it can shown that they don't each measure the other to be at a fixed distance. Hence they do not form a frame.

world line at the event of interest [10]. So, from moment to moment, the accelerated observer occupies a succession of MCIFs. We'll see shortly that this postulate of using MCIFs is to some extent validated experimentally.

Consider two identically accelerated observers in Figure 3, whose clocks have been synchronised in the inertial frame of the picture. As described in the caption, Blue and Red do not share a common standard of simultaneity, and hence do not form a frame. (It can also be shown that they each measure the other's distance to be increasing. Recall that a requirement for a frame is that its observers measure each other to be at fixed relative locations.)

Uniformly Accelerated Observers: By definition, these observers each feel a constant acceleration indefinitely. (This does not mean that each observer accelerates at a constant rate indefinitely; a uniformly accelerated observer's world line in an inertial frame is a hyperbola, and his speed asymptotes to the speed of light.) It turns out that a set of uniformly accelerated observers whose individual accelerations are set in just the right way by their distances from each other will *all* share a common standard of simultaneity. But if they want their clocks all to display the same value at the same time for them, the clocks to the right in the figure (which have lower accelerations than those to the left) must be "geared down in the factory" to tick slower than those at the left.

These observers are shown as the solid world lines in Figure 4, with the dashed lines being their lines of simultaneity. It can be shown that these observers each measure all of the other uniformly accelerated observers to be at fixed locations. The fact that they agree on simultaneity and are relatively fixed means that they form a frame, known as a *uniformly accelerated frame*. It is presumably the only other type of "large-scale" frame besides an inertial frame, although—unlike an inertial frame—the uniformly accelerated frame is not global: coordinates cannot be allocated outside the wedge of spacetime shown in Figure 4. In particular, the uniformly accelerated observers call the origin of the single space dimension in Figure 4 a "horizon", because the event at the origin of the figure's axes is simultaneous with all events in their frame, time as they perceive it

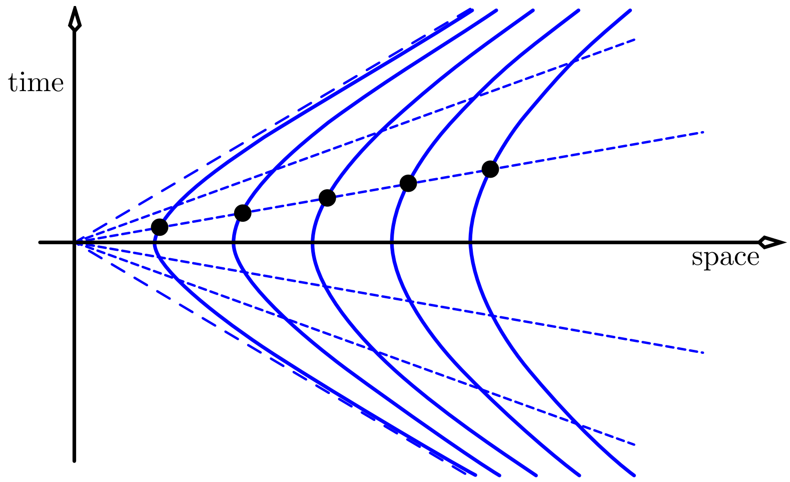

Figure 4: At all events marked with a black dot, each observer agrees that all other black-dot events occur at that moment. The observers' clocks tick at different rates, but these clocks can be geared in such a way that all observers can always say "When my clock displays 7 p.m., all other observers' clocks display 7 p.m." These observers also turn out each to measure the other to be at a fixed distance. Hence they form a frame: a "uniformly accelerated frame".

at this point has slowed to a stop and, to its left, that time runs backwards. But no information can ever reach them from this part of spacetime.

The uniformly accelerated frame with its "pseudo gravity" is the stepping stone, via Einstein's Equivalence Principle, to a consideration of real gravity. For example, mimicking the gearing-down of clocks to the right in the figure, GPS satellites are set to tick slightly slowly in the factory before they are placed in orbit. The success of GPS and classic experiments such as that performed by Pound, Rebka, and Snider can be taken as experimental validation of the MCIF postulate.

Observers in Rotational Motion: Consider a set of observers fixed to a disk that rotates in an inertial frame. This scenario has been discussed ever since the birth of special relativity, without yet finding a consensus among relativity specialists. Traditionally, analyses of simultaneity on the rotating disk treat a set of MCIFs of the observers fixed to its rim. They then apply a series of one-space-dimensional Lorentz transforms to conclude, after going once around the disk, that some events in the future of each observer are in some sense simultaneous with his present. This contradiction is noted, but seldom explored in any detail; the conclusion is simply that rotation does not produce a valid frame. This much is agreed upon by relativists and precise-timing specialists. I agree that rotation does not produce a valid frame, but I maintain that traditional analyses of this subject arrive at the right conclusion via an invalid use of MCIFs, since the one-space-dimensional Lorentz transform was never designed to analyse a "wrapping" of a single space dimension around the periphery of the two-space-dimensional rotating disk. If it was designed for that, then we would have to accept the idea that some events in the future of each observer are simultaneous with his present.

In reference [11], I have argued that to analyse a disk rotating in two dimensions, it is not relativistically valid to apply a one-space-dimensional analysis to create a helix

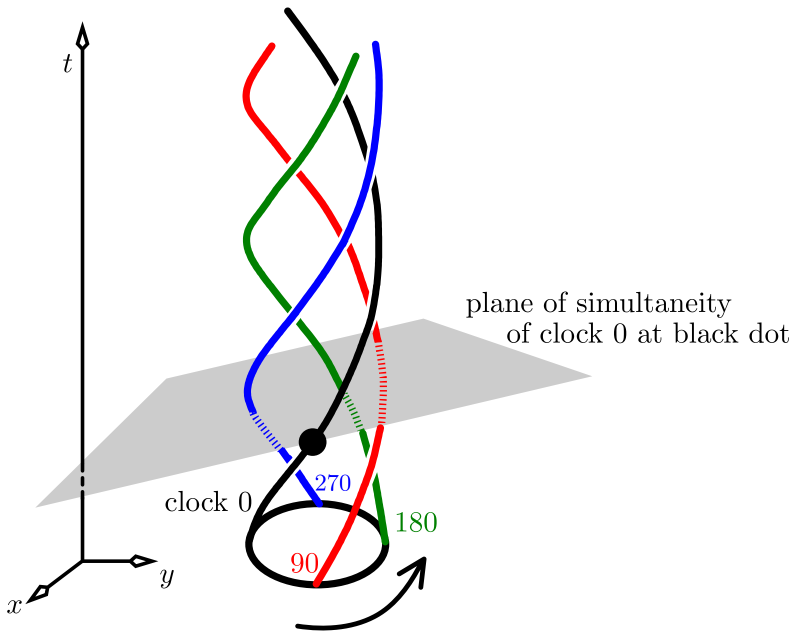

Figure 5: A set of helical world lines of four clocks on Earth's Equator, or on a rotating disk. The plane of simultaneity belongs to the event on the black world line marked by a black dot, as discussed in the paragraph after (A.11).

of simultaneity around the cylinder. Instead, we must apply the two-space-dimensional Lorentz transform to MCIFs, to construct the *plane* of simultaneity at any given event, using the same (conventional) approach as that followed in Appendix A for one space dimension to produce the line of simultaneity (A.11). This plane is determined from moment to moment by the latest MCIF. With this analysis, I find that observers fixed to the disk do not share a common standard of simultaneity, and hence they do not form a frame. No events in the future of any observer turn out to be simultaneous with his present.

This idea of creating a plane of simultaneity using the Lorentz transform is shown in Figure 5. There we see the helical world lines of four clocks fixed to the periphery of the disk that is rotating in the inertial frame of the figure. These could be interpreted as the world lines of four clocks on Earth's Equator, with no gravity to complicate the analysis. Suppose these four clocks have been synchronised in the ECI (which is thus the frame of Figure 5): the ECI says that at all times, the four clocks all display the same value. But the clocks themselves give different meanings to "now". Call the clock fixed at latitude 0 "clock 0" (whose world line is, say, black in Figure 5), and similarly for the clocks at latitudes 90° (red world line), 180° (green), and 270° (blue). We construct clock 0's plane of simultaneity at the event where it displays zero, and find the intersection events of this plane with the world lines of the other three clocks, noting their times at these intersection events. Recall the one-space-dimensional case in (A.11), where a line of simultaneity has a slope on a t-versus-x spacetime diagram of v, where v is the clock's velocity. In two space dimensions, the plane of simultaneity has the analogous tilt in spacetime; thus, as it extends a distance R (Earth's radius) to clock 90's world line, it rises along the time axis by vR (or vR/c^2 with c restored). The helical world lines of the comparatively slowly rotating clocks on Earth's Equator (speed 465 m/s in the ECI) are almost parallel to the t axis of Figure 5 [proof: see the analysis around (3.2)]. So this rise is

$$\frac{vR}{c^2} = \frac{465\ \mathrm{m/s} \times 6378\ \mathrm{km}}{9 \times 10^{16}\ \mathrm{m}^2/\mathrm{s}^2} \simeq 33\ \mathrm{ns}\,. \tag{2.1}$$

That is, when clock 0 displays zero, it says "At this moment, clock 90's displays 33 nanoseconds,[2] clock 180 displays zero, and (by symmetry) clock 270 displays −33 ns." But this reasoning applies around the circle. In particular, clocks 0 and 180 always agree that each displays the same time as the other,[3] but they disagree about which of clocks 90 and 270 displays the later time. This means they don't share a common standard of simultaneity, and hence the observers on the disk cannot constitute a fully relativistically bona-fide "rotating frame".

From the earliest days of special relativity, researchers have tried to analyse observers in rotational motion by making a rotational Galilei transform from an inertial frame. When treated as a possible change of frame, this use of a Galilei transform runs contrary to relativity: after all, central to special relativity is the replacement of the Galilei transform with the Lorentz transform, because the Lorentz transform produces coordinates obeying the relativistic notion of simultaneity. As early as 1922, it was recognised that a Galilei transform has no a-priori physical relevance to rotation [12], and yet even today, the transform is still being described by precise-timing practitioners as a relativistic change of frame: see, for example, equation (15) of [13] and also [14]. The transform *is* a change of coordinates, but that does not make it a change of frame. Instead, in this context the transform produces a set of what might be called rotating coordinates for the inertial frame in which the rotating system is rotating. These coordinates are not guaranteed to have any true physical relevance to the rotating system. In the context of Earth, they are a way of plastering ECI coordinates onto observers who are fixed in the ECEF, and this is certainly what is done to create our modern world's timing. But they are not true coordinates of a rotating frame. Some of this discussion is in line with that of Corum in [15]. Corum decries the use of the Galilei transform, but perhaps doesn't make the point that this transform is, at best, an attempt to create a set of rotating coordinates (not a rotating *frame*) in lieu of the fact that rotating frames don't exist. But this distinction that I am making here, of rotating coordinates versus a rotating frame, is probably completely unknown in the field of precise timing. See my further comments on this at the start of Section 6.

Observers on Earth: The oblate Earth with its gravity is a much more complicated example of rotational motion than the above disk. But because analyses of the disk have never lead to a consensus on, for example, the relativistic geometry of the disk, we should not expect the subject of precise timing on a rotating Earth to be in any advanced state. This is belied by the analyses found in many precise-timing papers, which wrongly assume (as pointed out above) that any arbitrary change of coordinates produces a new frame.

[2]More accurately, the helical shape of the world lines means that clock 90 turns out to be measured as located at approximately $(90 + 1.4 \times 10^{-10})°$. This measured skewing of clock positions around Earth's Equator is insignificant here.

[3]This phrase shows that although the typical times being discussed here, along with the form of (2.1), might suggest the Sagnac effect of Section 4.1, they are *not* the Sagnac effect, since the Sagnac effect involves time differences that increase linearly with longitude, and incorporates no relativity.

The calculation of (2.1) is significant, because it says that simultaneity in the ECEF fails at the level of tens of nanoseconds. This large time "discrepancy" does not affect the operation of GPS. The reason is that the calculations of GPS occur in the ECI. That is, GPS is based on the ECI time of *emission* of each satellite signal; it does not use the time at each satellite that the receiver calls "now" when it receives a signal. In other words, GPS does not use the time of the event at each satellite that is deemed by some observer to be simultaneous with the reception of the signal from that satellite at the receiver.

The definition of a frame as a set of observers who all share a common standard of simultaneity and measure no relative motion is well understood in classical mechanics, where the meaning of simultaneity is taken for granted in a non-relativistic way. But even in relativity textbooks, the definition of a frame is not commonly stated or explored. No doubt this is because special-relativity textbooks place almost all of their emphasis on inertial frames, so that the question of whether more complicated motion can produce a frame is virtually never addressed. Even the well-established subject of uniformly accelerated frames is given just a passing mention in most relativity texts—despite the fact that the uniformly accelerated frame is the springboard, via Einstein's Equivalence Principle, to a discussion of gravity in relativity. Referring to Pais's biography of Einstein [16], it seems that when Einstein first discussed acceleration in relativity, his aim was to make an immediate link to gravity. Modern writers have followed suit, using only very short discussions of acceleration to segue quickly into a discussion of gravity proper. I think that such abbreviated analyses bypass the richness and subtlety of uniformly accelerated motion as a subject in its own right that can shed light on other areas of relativity [9, 3].

2.3. Preliminary Comment on Including Gravity

The idea of simultaneity rests on a signal of constant speed being bounced from another clock. In non-inertial systems, light's speed is a function of its position. That turns out to offer no impediment to simultaneity being defined in the uniformly accelerated frame via MCIFs, although the details require some work, and rest on the geometry of the flat spacetime in which the uniformly accelerated frame resides. But when gravity is introduced to the picture, spacetime becomes curved, and the concept of simultaneity becomes far more problematic. We will take the position that the presence of gravity on a rotating Earth renders the lines and planes of simultaneity somewhat fuzzy. The result is that the numbers we produce in the following pages should be interpreted as optimistic estimates of quantities that are not especially well defined.

3. Simultaneity in the ECEF

In this section we examine the details of simultaneity relating to a set of clocks on Earth, and satellites orbiting Earth.

Equation (2.1) had the result that if clocks 0, 90, 180, and 270 all display the same time ("have the same age") in the ECI, then in the ECEF, clock 0 will say that in comparison to itself:

– clock 90 is 33 ns older,

– clock 180 is the same age, and
– clock 270 is 33 ns younger.

In particular, clock 180 will maintain that the ages of clocks 90 and 270 are the reverse of what clock 0 says they are. There is thus a 66 ns disagreement between clocks 0 and 180 about the ages of clocks 90 and 270. (And similarly, of course, there is a 66 ns disagreement between clocks 90 and 270 about the ages of clocks 0 and 180.) Suppose that these four clocks were replaced by satellites orbiting all in the same plane and spaced 90° apart. We require the analogous calculation to (2.1). The satellites are each at a distance of $R = 26{,}000$ km from Earth's centre. A satellite's speed in the ECI is $v = \sqrt{GM/R}$, where G is the gravitational constant and M is Earth's mass ($GM \simeq 4 \times 10^{14}$ SI units). Then, analogous to (2.1), we write

$$\frac{vR}{c^2} = \sqrt{\frac{GM}{R}} \times \frac{R}{c^2} = \frac{\sqrt{GMR}}{c^2} = \frac{\sqrt{4 \times 10^{14} \times 26 \times 10^6}}{(3 \times 10^8)^2} \simeq 1.1\ \mu\text{s}\,. \tag{3.1}$$

This calculation is only approximately applicable to GPS satellites, since these don't all orbit in the same plane. It's clear that from the viewpoint of one such satellite, the other satellites' clocks are mismatched by up to a microsecond. As discussed near the end of Section 2.2, this doesn't affect the operation of GPS because GPS satellites are synchronised in the ECI *and* the calculations that a GPS receiver runs to establish its position are ECI calculations. The fact that GPS satellites are not synchronised in the MCIF of any one of them is simply not relevant to the calculation performed by a receiver. The GPS receiver uses the ECI time of emission of each satellite's signal, and what the time is "now" at each satellite, whether in the frame of the receiver or of another satellite, is immaterial.

These values of 33 nanoseconds and 1.1 microseconds relate to the time displayed "now" on a distant clock. But more pertinent to two clocks attempting to synchronise is the extent to which they agree on the meaning of "now", since this might affect their ability to perform a data hand-shake as part of the synchronisation. To analyse this, we now extend the clock comparison in Figure 3 to two space dimensions to examine how well two clocks on Earth (say, at the same latitude) might be synchronised in the absence of gravity; that is, we calculate the analogous quantity to what might be called Figure 3's "synchronisation disagreement" of 7 p.m. − 5 p.m. = 2 hours. The two clocks at a single latitude on Earth can be treated as being on the rim of a disk in two spatial dimensions that rotates in the ECI. The lines of simultaneity in Figure 3 become the planes of simultaneity in Figure 5.

To begin, a big simplification can be made. Consider the helical world line of a single clock, say, on the Equator, as drawn in the frame of the distant stars over the course of one sidereal day in Figure 6. One sidereal day is a few minutes less than one solar day of 24 hours, but we can approximate one sidereal day as one solar day here. The height of the cylinder on which this world line is drawn is

$$\text{cylinder height} \simeq c \times 24 \text{ hours} \simeq 2.6 \times 10^{10} \text{ km}. \tag{3.2}$$

The width of the cylinder is

$$\text{cylinder diameter} = \text{Earth's diameter} \simeq 13{,}000 \text{ km}. \tag{3.3}$$

The cylinder thus has a height-to-diameter ratio of about 2 million. This allows us to approximate the helix as a straight line when analysing time increments of much less than one day—which is valid here, because (2.1) gives typical time increments of tens of nanoseconds

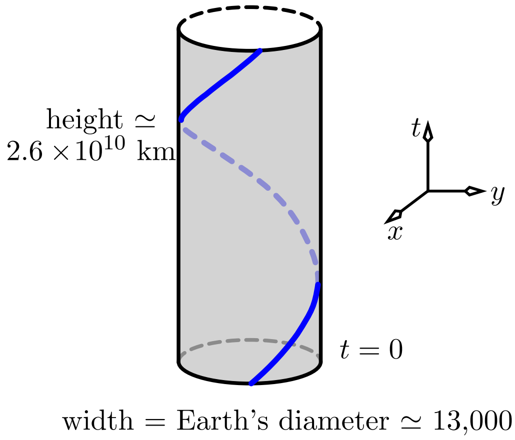

Figure 6: The (blue) world line of a clock on the Equator traces out a full revolution in one sidereal day. The cylinder surface is drawn merely to aid in the visualisation.

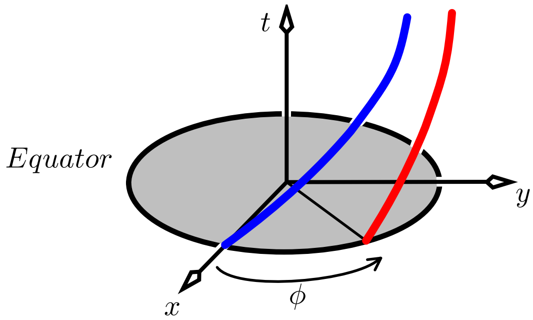

Figure 7: The world lines of two clocks on the Equator, which is modelled as a disk in the inertial frame of the distant stars

at most. (A related analysis that uses fully *helical* world lines appears in [11]; but it is far more complicated than the discussion here, and relies on an equation resembling the Kepler equation of orbital theory, which cannot be solved in terms of standard functions.)

Now consider two clocks, "Blue" and "Red", fixed to the Equator. The blue clock is at longitude 0, and the red clock is at longitude ϕ. Parts of their world lines spanning a small time interval are drawn in Figure 7 around the time $t = 0$, in the inertial frame of the distant stars. At $t = 0$, Blue is on the x axis, and its position and this time define an event A (analogous to event A in Figure 3). At this event A we will do the following, as shown in Figure 8:

1. Construct the blue plane of simultaneity of Blue (analogous to the blue dashed line in Figure 3).

2. Find the event B where this blue plane intersects the red world line of Red (analogous to event B in Figure 3).

3. Construct the red plane of simultaneity of Red at event B (analogous to the red dashed line in Figure 3).

4. Find the event C where this red plane intersects the blue world line of Blue (analogous to event C in Figure 3).

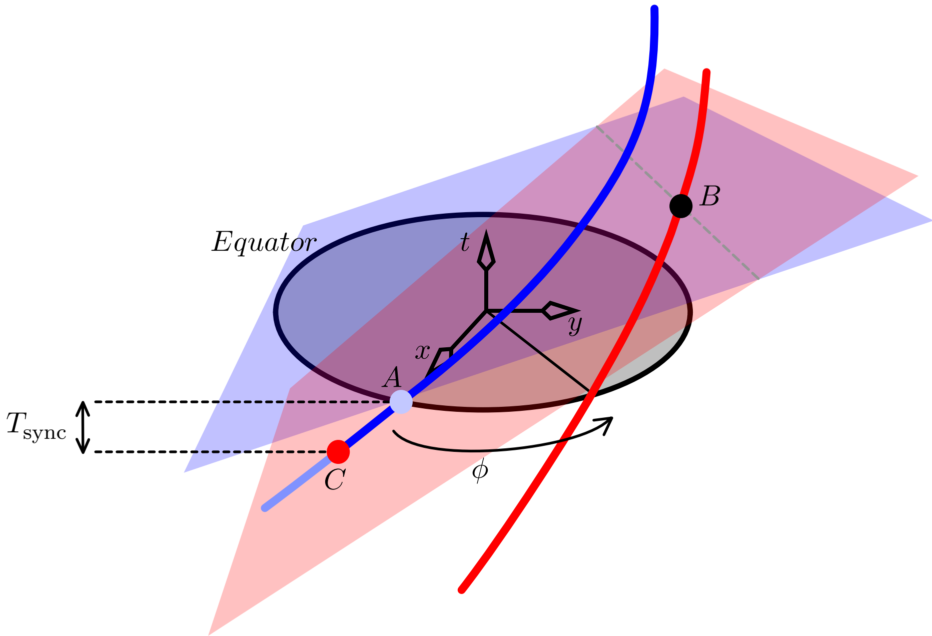

Figure 8: The two-space-dimensions version of Figure 3. The blue and red dashed lines in Figure 3 have here become blue and red planes, respectively.

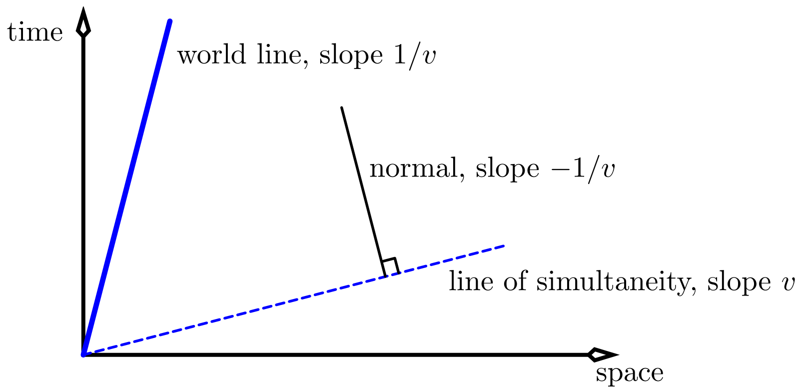

Figure 9: In one space dimension, the slopes of the world line, the line of simultaneity, and the normal to that line of simultaneity are written in terms of the velocity v of the clock on the world line

The difference between the times of events A and C in the inertial frame defines the extent to which the two clocks can agree on simultaneity. We will calculate this difference in times for several values of clock separation ϕ, and also for some satellites, which can be treated in the same formalism as tracing out helical world lines of a larger radius than that of Earth, and whose world lines can also be approximated as straight for the relevant time increments here.

In what follows, we will use a geometrical view of spacetime that uses the standard 3-component formalism of vectors in three *spatial* dimensions. That is, we will order the components of our coordinate vectors as (x, y, t), because the t axis here takes the place of the z axis in the familiar analyses of 3-space. The basic tool that we use and build on is the one-space-dimension picture in Figure 9. The line of simultaneity in that figure is where the plane of simultaneity in two spatial dimensions cuts the ty plane at $t = 0$. Using that idea, start with event A, which has coordinates

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_A = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}. \tag{3.4}$$

Making no distinction between row and column arrays for our coordinate vectors, we have

$$\text{normal to blue plane of simultaneity at } A = (0, 1, -1/v)\,. \tag{3.5}$$

The blue plane is the set of the following events:

$$\text{blue plane:} \quad t = vy\,, \quad x = \text{anything}. \tag{3.6}$$

We must find B, where this blue plane intersects the red world line. If the Equator has radius R, the red world line is the set of events

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_{\text{red}} = \begin{bmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{bmatrix} + \lambda \boldsymbol{n}_{\text{red}}\,, \tag{3.7}$$

where the parameter λ takes on all real values, and $\boldsymbol{n}_{\text{red}}$ is any direction vector of the red world line. In particular, $\boldsymbol{n}_{\text{red}}$ is found by rotating any direction vector of the blue world line ($\boldsymbol{n}_{\text{blue}}$) through angle ϕ right-handed around the t axis. Start with

$$\boldsymbol{n}_{\text{blue}} = (0, 1, 1/v)\,. \tag{3.8}$$

Now write

$$s \equiv \sin\phi\,, \quad c \equiv \cos\phi\,. \tag{3.9}$$

The corresponding direction vector of the red world line is then

$$\boldsymbol{n}_{\text{red}} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{n}_{\text{blue}} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1/v \end{bmatrix} = \begin{bmatrix} -s \\ c \\ 1/v \end{bmatrix}. \tag{3.10}$$

Hence, from (3.7), the red world line has equation

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_{\text{red}} = \begin{bmatrix} Rc \\ Rs \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -s \\ c \\ 1/v \end{bmatrix}. \tag{3.11}$$

Where is this cut by the blue plane (3.6)? The relation $t = vy$ gives, from (3.11),

$$\lambda/v = v(Rs + \lambda c)\,, \quad \text{in which case } \lambda = \frac{Rs}{1/v^2 - c} \text{ at event } B. \tag{3.12}$$

Event B thus has coordinates, from (3.11),

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_B = \begin{bmatrix} Rc \\ Rs \\ 0 \end{bmatrix} + \frac{Rs}{1/v^2 - c} \begin{bmatrix} -s \\ c \\ 1/v \end{bmatrix} = \frac{R}{1 - cv^2} \begin{bmatrix} c - v^2 \\ s \\ sv \end{bmatrix}. \tag{3.13}$$

Now we require the red plane of simultaneity at B. The normal to this plane is the rotated version of (3.5):

$$\text{normal to red plane of simultaneity at } B = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1/v \end{bmatrix} = \begin{bmatrix} -s \\ c \\ -1/v \end{bmatrix}. \tag{3.14}$$

This red plane thus has equation

$$-sx + cy - t/v = \alpha \text{ (a constant)}. \tag{3.15}$$

The constant α is found by noting that event B lies on this red plane. Specifically, (3.15) combines with (3.13) to give

$$\alpha = \frac{R}{1 - cv^2}\big[- s(c - v^2) + cs - sv/v\big] = \frac{Rs(v^2 - 1)}{1 - cv^2}. \tag{3.16}$$

Combining (3.15) and (3.16), the red plane of simultaneity through B has equation

$$\text{red plane:} \quad -sx + cy - t/v = \frac{Rs(v^2 - 1)}{1 - cv^2}. \tag{3.17}$$

We now intersect this red plane with the blue world line to find event C. Using (3.8), the blue world line has equation

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_{\text{blue}} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 1 \\ 1/v \end{bmatrix} \tag{3.18}$$

for a parameter λ_1 that takes on all real values. Intersect this with the red plane (3.17) to give event C:

$$-sR + c\lambda_1 - \lambda_1/v^2 = \frac{Rs(v^2 - 1)}{1 - cv^2} \quad \text{at event } C. \tag{3.19}$$

It follows that

$$\lambda_1 = \frac{Rs(c - 1)}{(1/v^2 - c)^2} \quad \text{at event } C. \tag{3.20}$$

Hence, (3.18) gives event C's coordinates as

$$\begin{bmatrix} x \\ y \\ t \end{bmatrix}_C = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + \frac{Rs(c - 1)}{(1/v^2 - c)^2} \begin{bmatrix} 0 \\ 1 \\ 1/v \end{bmatrix}. \tag{3.21}$$

In particular, the time of event C in the inertial frame is

$$t_C = \frac{Rs(c - 1)}{(1/v^2 - c)^2 v}. \tag{3.22}$$

Clearly, for $v > 0$ (which corresponds to Earth's natural spin) and $\phi \leqslant 180°$, $t_C \leqslant 0$. Note too that $t_A = 0$. The difference between the times of events A and C for $\phi < 180°$ is called T_{sync} in Figure 8:

$$T_{\text{sync}} \equiv t_A - t_C = \frac{R \sin\phi\,(1 - \cos\phi)}{(1/v^2 - \cos\phi)^2\, v}. \tag{3.23}$$

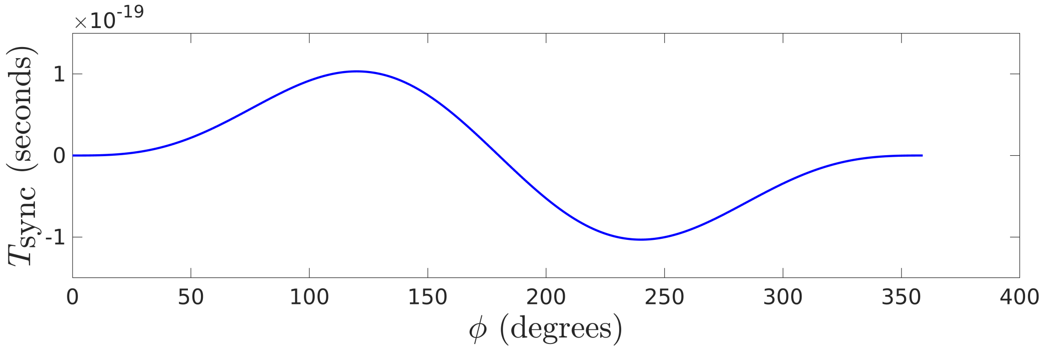

Figure 10: T_{sync} versus disk angle ϕ (or longitude on Earth), from (3.25)

T_{sync} quantifies the fundamental disagreement in simultaneity or synchronisation for clocks that are a longitude ϕ apart, fixed to the edge of a rotating disk of radius R, and whose speed (in the inertial frame in which the disk's centre is at rest) is much less than the speed of light, because we approximated the blue and red world lines as straight.

For Earth's Equator, $v \simeq 465\ \text{m s}^{-1}/(3 \times 10^8\ \text{m s}^{-1}) \simeq 1.6 \times 10^{-6} \ll 1$. So write

$$1/v^2 - \cos\phi \simeq 1/v^2\,. \tag{3.24}$$

Then (3.23) becomes

$$T_{\text{sync}} \simeq Rv^3 \sin\phi\,(1-\cos\phi)\,. \tag{3.25}$$

We wish to convert (3.25) to use conventional dimensions in its input and output. From now, drop the use of the "s, c" shorthand of (3.9), and denote the speed of light by c. To convert (3.25) to conventional dimensions, divide it by c^4:

$$T_{\text{sync}} \simeq \frac{Rv^3}{c^4} \sin\phi\,(1-\cos\phi)\,, \quad \text{where all variables have conventional dimensions.} \tag{3.26}$$

A plot of T_{sync} versus longitude ϕ is shown in Figure 10, using a value of $R = 6400$ km (Earth's radius). The maximum value of around 10^{-19} seconds occurs at $\phi = 120°$, and it drops to zero very quickly as ϕ tends to zero. Indeed, using the small-ϕ approximations

$$\sin\phi \simeq \phi\,, \quad \cos\phi \simeq 1 - \phi^2/2\,, \tag{3.27}$$

it's clear that $T_{\text{sync}} \propto \phi^3$ for small ϕ. Such small values of T_{sync} lie far beyond the accuracy of current communications technology, so we conclude that a mismatch in the meaning of "now" will have no effect on any handshakes currently made between clocks on Earth.

What is the value of T_{sync} for clocks that are on satellites, each at a distance R from Earth's centre? Circular motion is sufficient to analyse here, in which case a satellite's speed in the ECI is $v = \sqrt{GM/R}$ [which is far less than the speed of light, so (3.24) still holds]; so substitute that value of v into (3.26) to give

$$T_{\text{sync}} \simeq \frac{(GM)^{3/2} \sin\phi\,(1-\cos\phi)}{\sqrt{R}\,c^4}\,. \tag{3.28}$$

GPS satellites have $R \simeq 26{,}000$ km. Use $GM \simeq 4 \times 10^{14}$ SI units and choose the worst-case value of $\phi = 120°$. Then,

$$T_{\text{sync}} \simeq 2.5 \times 10^{-16}\ \text{s.} \tag{3.29}$$

For low Earth-orbit satellites ($R \simeq 7000$ km), a similar calculation gives $T_{\text{sync}} \simeq 5 \times 10^{-16}$ s.

The above instances of T_{sync} say that clocks fixed to Earth's surface or on satellites have a very small mismatch in what they say "is happening now", and this presumably sets a limit to the efficacy of a handshake between their clocks to attempt a synchronisation procedure. But this analysis should not be construed as saying that these observers can agree on the times of *all* events—even those occurring on Earth—to this accuracy. For example, for "clock 0" and "clock 180" that lie on opposite sides of the Equator ($\phi = 180°$), equation (3.26) says that $T_{\text{sync}} = 0$. [This value is easily seen without any mathematics, because the plane of simultaneity of clock 0 at an ECI time of $t = 0$ in Figure 5 will intersect the world line of clock 180 at the same ECI time of $t = 0$, and vice versa.] This means that events A and C in Figure 8 coincide for clocks 0 and 180. Hence those clocks certainly agree on the meaning of "now"; but they do *not* agree on the time displayed on a clock that is fixed at, say, $\phi = 90°$. Indeed, as mentioned just after (2.1) and again just before (3.1), clocks 0 and 180 will disagree on the age of clock 90 by 66 nanoseconds. As stated earlier, there is nothing that can be done to "fix" this; it is a result of relativity. So even though T_{sync} is exactly zero for clocks 0 and 180, that only means that they agree on the meaning of "now"; but they disagree at the level of tens of nanoseconds on the simultaneity of events that are some distance from both of them. As pointed out near the end of Section 2.2 and also just after equation (3.1), this has no effect on the operation of GPS.

4. Details of Synchronising Clocks

It's important to appreciate that because a set of clocks that is rotating in an inertial frame cannot be synchronised such that they all allocate the same time coordinate to an event that they all agree to be happening "now", a rotating system is not a true frame in relativity, and so constructing coordinates for it is problematic [11]. So, although the ECEF is not a true frame, in practice such a set of rotating observers is labelled with the coordinates of a well-defined frame such as the ECI. This is in fact what is done in practice. Here are two ways in which two clocks at rest in the ECEF might be synchronised to ECI time, excluding any discussion of gravity for now.

An external master clock is used. The clocks both receive a single signal from a master clock based on a satellite. Each notes their own time of reception of the signal. Knowing their own positions and the satellite's position gives all the necessary timing information that allows them to synchronise with the master clock's ECI time. In the world of GPS, this is called "common-view time transfer".

One clock synchronises the other. This is a more subtle procedure, because the clock doing the synchronising must use a time (ECI) that is not its own "proper time". This method of synchronisation might be the only option in the event that common-view GPS signals are not available. If the clock doing the synchronising does attempt to synchronise as best it can with another clock to an "ECEF time", it must be aware that if it communicates with the other clock using, say, light, then that light is not necessarily travelling at speed c, because the synchronisation is being performed in a non-inertial frame.

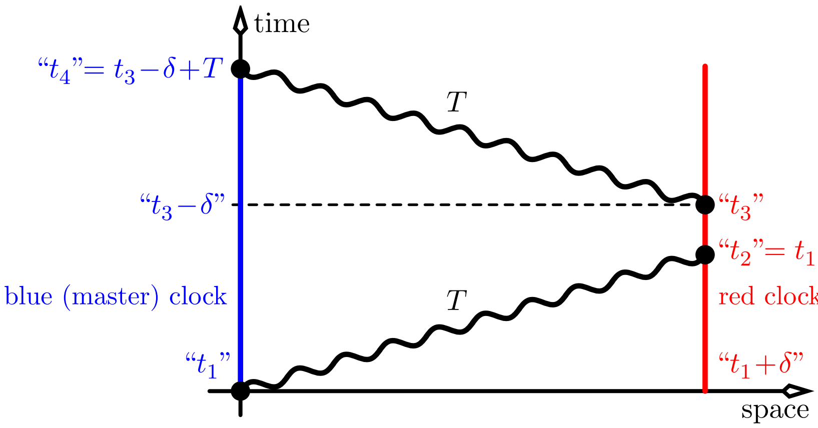

Figure 11: Synchronising relatively stationary clocks in a frame in which light's speed is independent of direction. The blue clock exchanges a light pulse with the red clock to determine the value of δ, the unknown amount by which Red's display is initially ahead of Blue's display. The light pulse takes an unknown time T to travel in each direction. The quantities in quotes are the readings on the clocks—not the frame's time of those events.

In the next few pages, we describe standard procedures for synchronising two clocks to the time of an inertial frame (in which they may or may not be at rest). Known in the timing world as the IEEE 1588 standard, it is really just a practical way of implementing the procedure described back in Figure 1. Consider first two clocks in an inertial frame, that wish to synchronise with each other so that their time displays give a fully valid time from a relativistic point of view. Figure 11 shows a plot of time versus separation of these clocks in their inertial frame, in which light's speed is independent of its direction of motion. The clocks are separated by an unknown but fixed distance. The blue clock is the master, and we require to set the red clock to display the same time as the blue. Initially the clocks might display different numbers: we suppose that Red is always ahead of Blue by an unknown amount δ, which we require to measure: subtracting δ from Red's display will accomplish the synchronisation. When Blue displays a time t_1, it sends a light pulse to Red. This transmission takes some unknown time T. The light pulse reaches Red when that clock displays a time t_2. At some later time when Red displays t_3, Red sends a light pulse back to Blue, which arrives when Blue displays t_4. All readings t_1, t_2, t_3, and t_4 are recorded. For convenience, define two numbers α and β from the measured readings:

$$\begin{aligned} \alpha &\equiv t_2 - t_1 = \quad \delta + T\,, \\ \beta &\equiv t_4 - t_3 = -\delta + T\,. \end{aligned} \tag{4.1}$$

These equations invert to give

$$T = \frac{\alpha+\beta}{2}\,, \qquad \delta = \frac{\alpha-\beta}{2}\,. \tag{4.2}$$

The offset δ is now subtracted from Red's display to accomplish the synchronisation. We have the added bonus of measuring the clocks' separation cT.

The IEEE 1588 procedure assumes the speed of light is independent of direction. The rotating Earth is not inertial; as discussed next, the speed of light in the ECEF does depend

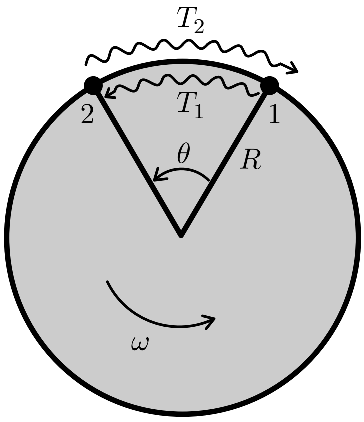

Figure 12: The Sagnac effect. The spinning disk above is shown in an inertial frame. Each black disk is a clock.

on direction. But as discussed earlier in Section 4, we can choose to synchronise clocks on a rotating Earth in the inertial ECI frame. In that case, we *can* use IEEE 1588, provided we take Earth's rotation into account. We discuss this in the next sections.

4.1. The Sagnac Effect

The *Sagnac effect* results from sending two light pulses in opposite directions around a loop that is rotating in, say, an inertial frame, and measuring the flight time of each pulse. In the inertial frame, the two pulses travel at the same speed through different distances to the emitter/receiver, and so one of them will complete its trip before the other. In the non-inertial "frame" in which the loop is at rest, both pulses travel the same path and hence the same distance; but because one pulse completes its trip before the other, we conclude that they have different speeds. No relativity is required to derive the Sagnac effect.

Suppose clocks 1 and 2 in Figure 12 (shown as small black disks) are fixed to a spinning disk, viewed from an inertial frame. The disk has radius R and rotates with angular speed ω in the inertial frame, in which case each clock has velocity $v = R\omega$ in the inertial frame. The clocks are a distance apart along the rim of $D = R\theta$. The inertial frame measures a time T_1 for the signal emitted by 1 to be received by 2, and likewise a time T_2 for the signal emitted by 2 to be received by 1. Suppose the pulses travel in a vacuum, and hence both have speed c in the inertial frame of the figure.

Consider the pulse sent from clock 1 to clock 2. The distance it travels equals the fixed distance around the disk between the clocks plus the extra distance that clock 2 has moved away from the pulse during the transit time:

$$\underbrace{cT_1}_{\substack{\text{total distance}\\ \text{light travels}\\ \text{in inertial}\\ \text{frame}}} = \underbrace{D}_{\substack{\text{disk}\\ \text{separation}\\ \text{of clocks}}} + \underbrace{vT_1}_{\substack{\text{distance clock 2}\\ \text{recedes during}\\ \text{transit time}}} . \tag{4.3}$$

It follows that

$$T_1 = \frac{D}{c - v} . \tag{4.4}$$

When $v \ll c$, (4.4) becomes

$$T_1 = \frac{D}{c} \times \frac{1}{1 - v/c} \simeq \frac{D(1 + v/c)}{c} = \frac{D}{c} + \frac{Dv}{c^2}\,. \tag{4.5}$$

For the opposing pulse sent from clock 2 to clock 1, the distance that the pulse travels is shorter than the fixed distance around the disk between the clocks by the amount that clock 1 moves toward the pulse during the transit time:

$$\underbrace{cT_2}_{\substack{\text{total distance}\\ \text{light travels}\\ \text{in inertial}\\ \text{frame}}} = \underbrace{D}_{\substack{\text{disk}\\ \text{separation}\\ \text{of clocks}}} - \underbrace{vT_2}_{\substack{\text{distance clock 1}\\ \text{approaches during}\\ \text{transit time}}}. \tag{4.6}$$

This leads to

$$T_2 = \frac{D}{c + v}\,. \tag{4.7}$$

When $v \ll c$, this becomes

$$T_2 \simeq \frac{D}{c} - \frac{Dv}{c^2}\,. \tag{4.8}$$

The difference between these two transit times is

$$T_1 - T_2 \simeq 2Dv/c^2. \tag{4.9}$$

The fact that $T_1 \neq T_2$ is called the *Sagnac effect*. It is not a relativistic effect, since the above derivation would be valid even if relativity had never been discovered—we are simply using the fact that light has some speed denoted c in the ECI.[4]

Suppose the observers are co-located on Earth's Equator and exchanging light pulses "the long way around". Then,

$$R = 6378 \text{ km}, \quad v = 465 \text{ m/s}, \quad \theta = 2\pi\,. \tag{4.10}$$

Hence (4.5) and (4.8) give

$$\left\{ \begin{matrix} T_1 \\ T_2 \end{matrix} \right\} (\text{Earth's Equator}) \simeq \frac{2\pi R}{c} \pm \frac{2\pi R v}{c^2} \simeq \frac{2\pi R}{c} \pm 207 \text{ ns}. \tag{4.11}$$

In the ECI, a pulse of light sent east around the world on the Equator (see T_1) will take 207 ns longer to reach its point of origin than we would have expected had we thought that Earth was not spinning. (Clocks on Earth are in a gravity field and are moving in the ECI, but the relativistic slowing that results is negligible compared to the value of 207 ns.) The Sagnac effect can thus be used to measure an object's spin rate, making it useful in modern inertial navigation systems.

An observer who is also at rest on the disk will conclude that, because clocks 1 and 2 have a fixed separation and one pulse completes a full circuit before the other, the speed of light must depend on its direction of travel around the disk.[5] There is no problem with this, of

[4]This disproves the comment in [17] that says the presence of c^2 means a calculation incorporates relativity.

[5]From the viewpoint of this observer's accelerated frame, light slows when it moves counter to the direction of perceived motion of the distant stars, and light speeds up when it moves *with* the distant stars. This is entirely normal: the same effect arises when we imagine spinning around on the spot, and asking what is happening to light rays that are travelling across the Moon's surface. From our rotating point of view, they (along with the Moon itself) are travelling much faster than c. But when measured *locally* (that is, by an observer who is standing on the Moon), they move with the usual speed c.

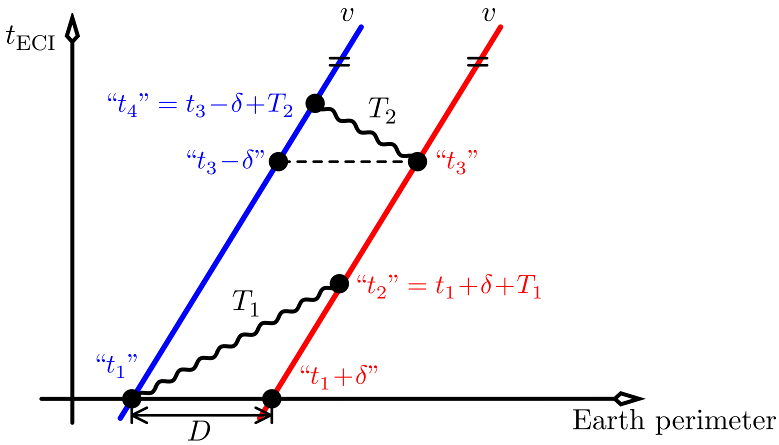

Figure 13: The modification of Figure 11 to synchronise Earth-fixed clocks to ECI time

course, since an observer fixed to the disk is not inertial, and so relativity does not prescribe light's speed to be c when measured by that observer. But we must not assume that an ECEF observer says that light's speed is $c - v$ east [recalling (4.4)] and $c + v$ west [(4.7)], because ECEF observers have a non-trivially evolving plane of simultaneity (recall Figure 5). Indeed, the rotating-disk analysis of [11] suggests that these speeds $c \mp v$ are only *averages* in each direction for trips around the entire planet. That reference focusses on the position of the plane of simultaneity drawn in Figure 5, and calculates where this plane intersects the helical world lines of the relevant clocks and light pulses to answer the question "Where are the light pulses now?" from moment to moment. What emerges from that non-trivial analysis is that the standard Sagnac calculations presented above in the ECI give a form of average of a set of speeds pertinent to the ECEF. For example, from the viewpoint of an ECEF observer, the speed of an east-bound pulse in the small-v limit turns out to be c at the location of the observer, and $c - 2v$ on the other side of the disk: compare these two speeds with the ECI Sagnac value of $c - v$ in (4.4). The fact that the analysis in [11] says that light will be measured to have speed c in *any* direction in a laboratory experiment on Earth is just as well, since confusing ECEF and ECI in the Sagnac analysis would lead us to expect that such a measured speed will be anywhere in the range $c - v$ to $c + v$ depending on its direction of travel, where $v \simeq 465$ m/s. In 1972, the speed of light was measured in the laboratory to be 299,792,456.2 ± 1.1 m/s [18], which is within 1 m/s of the standard value $c =$ 299,792,458 m/s. This measurement supports the analysis in [11].

4.2. Synchronising Earth-Fixed Clocks in the ECI

Because clocks cannot be synchronised in the ECEF, we settle for something less: that they be synchronised in the ECI. This is done by modifying Figure 11 to incorporate the Sagnac effect. Refer to Figure 13, which is an ECI representation of two clocks, at (say) the same latitude on Earth, exchanging timing signals in the manner of Figure 11. Because the clocks are drawn in the ECI, they have speed v due to Earth's rotation. Appendix C.3 shows that light moves over Earth's surface with an ECI speed of approximately $c + v^2/(2c)$, which we approximate by c since $v \ll c$. The clocks are a distance D apart, as shown in Figure 12. This

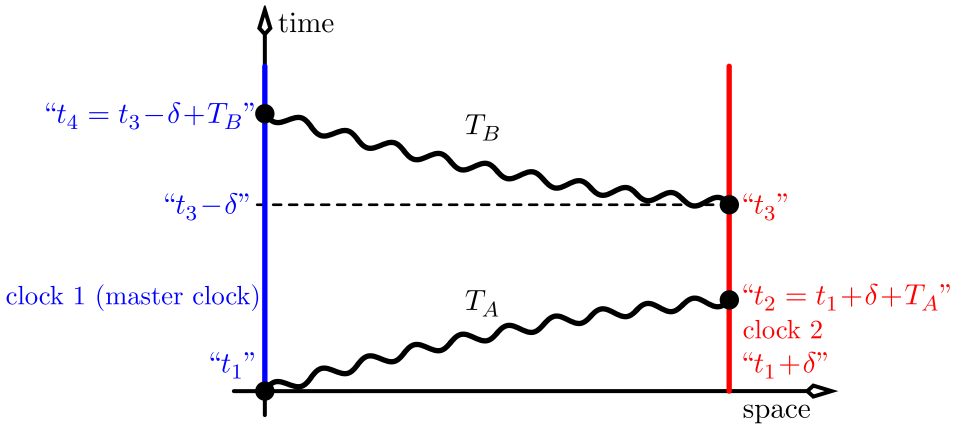

Figure 14: A rotating disk isn't inertial, and so for an observer on the disk, light's speed depends on its direction. Thus, compared to the inertial setup of Figure 11, we allow for different transmission times T_A, T_B, of the light pulses. The pulses themselves might have a position-dependent speed, so their world lines are drawn curved.

setup allows us to use the results of Section 4.1. Equation (4.1) is then modified to

$$\begin{aligned}\alpha \equiv t_2 - t_1 &= \quad \delta + T_1 \overset{(4.4)}{=\!=} \quad \delta + D/(c-v)\,,\\ \beta \equiv t_4 - t_3 &= -\delta + T_2 \overset{(4.7)}{=\!=} -\delta + D/(c+v)\,.\end{aligned} \tag{4.12}$$

These equations invert to give

$$D = \frac{(\alpha+\beta)(c^2-v^2)}{2c}\,, \qquad \delta = \frac{\alpha-\beta}{2} - \frac{v(\alpha+\beta)}{2c}\,. \tag{4.13}$$

Compare this with (4.2), which applies to synchronising clocks at rest in a shared inertial frame: those clocks agree that they display the same time. The clocks in Figure 13 will not agree that they display the same time, because they are being synchronised to the time of a frame in which they are moving. Nonetheless, when $v = 0$, their frame becomes the ECI, and so it makes sense that the zero-v limit of δ in (4.13) is (4.2). The Sagnac effect is not a standard part of the IEEE 1588 procedure, and it must be implemented in the way of (4.13) if such accuracy is required.

Attempting to Synchronise within the ECEF

It must be stressed that the above procedure in Figure 13 synchronises clocks in the ECI, not the ECEF. Although clocks cannot be synchronised in the ECEF, for the purpose of a lab experiment it might well be that two clocks are required to agree with each other's time in an approximation to a "local piece" of the ECEF. The clocks are now defined to be at rest, and it might then be thought that the Sagnac effect could be incorporated into the calculation of δ in Figure 11 in the following way. First, Figure 11 might be redrawn as Figure 14, where the outbound and inbound signals are now not assumed to have the same speed (in accordance with Sagnac in the ECEF). The analogue of (4.1) is then

$$\begin{aligned}\alpha \equiv t_2 - t_1 &= \quad \delta + T_A\,,\\ \beta \equiv t_4 - t_3 &= -\delta + T_B\,.\end{aligned} \tag{4.14}$$

Solving these equations for δ gives

$$\delta = \frac{\alpha - \beta + T_B - T_A}{2} \,. \tag{4.15}$$

What is $T_B - T_A$? We would be applying relativity wrongly if we thought that it differed from $T_2 - T_1$ in (4.9) merely by a time-dilation factor of $\gamma = 1/\sqrt{1 - v^2/c^2}$ (which is approximately $1 + 10^{-12}$ on the Equator, so is negligible for all current purposes). This is because we have not made allowance for an "out-of-synchrony" amount similar to (2.1), which is tens of nanoseconds. But we can barely do even that: even drawing Figure 14 becomes problematic, because its standard time–space setup is tied closely to a global standard of simultaneity, which simply does not exist for rotating observers.

We might try to make do as best we can with an analysis that makes no reference to the ECI, and assumes that the ECEF is somewhat well defined. To that end, we might borrow from (4.4) and (4.7) to write

$$\begin{aligned} T_A &= \frac{D}{\text{average speed of eastbound light in ECEF}} \,, \\ T_B &= \frac{D}{\text{average speed of westbound light in ECEF}} \,. \end{aligned} \tag{4.16}$$

But we should not assume that the eastbound and westbound speeds of light in the ECEF are $c - v$ and $c + v$ respectively. Recall the discussion following (4.11), which refers to the work in [11]. We will calculate $T_A - T_B$, which then compares directly with $T_1 - T_2$ in (4.9). Implementing the full expressions in [11] for the speed of light as measured by an observer fixed to a rotating disk is difficult, since they use an angular coordinate that is not trivially related to longitude on a disk or on Earth. But when $|v| \ll c$, equation (44) of [11] says that the light speeds on the disk are approximately

$$\text{speed of light in ECEF on Equator} \simeq c \mp v\,(1 - \cos\theta) \quad \begin{cases} \text{east} \\ \text{west.} \end{cases} \tag{4.17}$$

The angular coordinate θ has a difficult interpretation, but it is approximately $\theta \simeq D/R$. In that case, when the clocks are separated by a distance D that is somewhat less than the disk radius R, we can use $\cos\theta \simeq 1 - \theta^2/2$ to write (4.17) as

$$\text{speed of light} \simeq c \mp \frac{vD^2}{2R^2} \equiv c \mp \alpha \quad \begin{cases} \text{east} \\ \text{west,} \end{cases} \tag{4.18}$$

where $\alpha \equiv vD^2/(2R^2) \ll c$. We might then use (4.16) to write

$$T_A - T_B \simeq \frac{D}{c - \alpha} - \frac{D}{c + \alpha} \simeq \frac{2D\alpha}{c^2} = \frac{vD^3}{c^2 R^2} \,. \tag{4.19}$$

Compare this to $T_1 - T_2$ in (4.9):

$$\frac{T_A - T_B}{T_1 - T_2} = \frac{D^2}{2R^2} \ll 1 \text{ inside a lab.} \tag{4.20}$$

Table 1 shows comparison values of $T_A - T_B$ and $T_1 - T_2$ for some clock separations D of interest, on Earth's Equator. The point here is that if we do attempt to synchronise two

Table 1: Comparison of values of $T_A - T_B$ and $T_1 - T_2$ for some clock separations D of interest on Earth's Equator, using $R = 6378$ km and $v = 465$ m/s. Strictly speaking, the analysis attempts to be applicable to a "small" region only; so, in practice, D should perhaps not be chosen as large as the tens or hundreds of kilometres in the table

D:	D	1 km	10 km	100 km	1000 km
$T_A - T_B$ (ns):	$vD^3/(c^2R^2)$	1.3×10^{-10}	1.3×10^{-7}	1.3×10^{-4}	0.13
$T_1 - T_2$ (ns):	$2vD/c^2$	0.0103	0.103	1.03	10.3

clocks to some form of local ECEF time using the IEEE 1588 procedure, the "Sagnac offset" $(T_B - T_A)/2$ in (4.15) is *far* smaller than what would result if the incorrect expression $(T_2 - T_1)/2$ was used instead. For example, the table says that over a kilometre, the standard Sagnac offset is about 10 picoseconds, and the above analysis says that this offset should *not* be applied to any synchronisation procedure.

Transmitting Light through a Fibre

Equations (4.5) and (4.8) give the correction $\pm Dv/c^2$ needed due to Earth's rotation when transmitting timing information with a light pulse. In fact, this correction is unchanged to first order in v/c if the light is transmitted through a fibre.

This can be seen as follows. Suppose that light travels through a fibre in the ECEF at a speed of c^F, where "F" mean fibre. Refer to Figure 12, and say that the light travelling from clock 1 to clock 2 (taking time T_1) travels at speed c_1^I in the inertial frame of that figure ("I" stands for inertial frame). Light travelling from clock 2 to clock 1 (taking time T_2) travels at speed c_2^I. These speeds need not equal c because the light is travelling through a fibre. Equations (4.4) and (4.7) are modified to

$$T_1 = \frac{D}{c_1^I - v}, \quad T_2 = \frac{D}{c_2^I + v}. \tag{4.21}$$

How are c_1^I and c_2^I related to c^F? It's not clear to what extent adding velocities in the usual relativistic fashion is valid when rotation is involved, given that the circumference of the rotating disk is not contracted in the inertial frame. But we suppose that this question introduces an uncertainty at the level of a factor of $\gamma = 1/\sqrt{1 - v^2/c^2}$. In that case, denoting relativistic velocity addition by $\oplus$ leads to

$$c_1^I = c^F \oplus v = \frac{c^F + v}{1 + c^F v/c^2}, \quad -c_2^I = -c^F \oplus v = \frac{-c^F + v}{1 - c^F v/c^2}. \tag{4.22}$$

To first order in v/c, these become

$$c_1^I \simeq c^F + v - c^{F^2} v/c^2, \quad c_2^I \simeq c^F - v + c^{F^2} v/c^2. \tag{4.23}$$

Placing these into (4.21) and keeping to first order in v/c finally produces

$$T_1 - T_2 \simeq 2Dv/c^2. \tag{4.24}$$

This is identical to (4.9). This result is well known in devices that use the Sagnac effect to detect rotation, by running light through optical fibre. But at an ultra fine level of accuracy it should be questioned, because the relativistic addition of velocities used above is only valid for inertial frames, whereas the spinning devices used in Sagnac accelerometers are not inertial; for example, the relevant lengths in these devices are not Lorentz contracted, and hence don't obey the usual simple rules of relativistic addition of velocities. No well-known work has been done in this difficult and specialised area of relativity.

4.3. Carrying a Clock on a Rotating Earth

Suppose we have a set of clocks that are fixed to the rim of a disk that rotates in an inertial frame, and they are synchronised in this frame (say, the ECI). We now carry one of them to the location of another clock and compare their times. They will no longer agree because their average speeds in the ECI have been different. By how much do their displays differ? Note that the following analysis is necessarily relativistic. Despite this, it is often confused with the Sagnac effect in precise-timing literature, as discussed in Appendix B.1.

Suppose the disk rotates "eastward" in the ECI such that the clocks all have velocity $v > 0$ in the ECI. Initially, all display zero in the ECI. One of these clocks ("clock 1") is moved with velocity V relative to the disk through a displacement D on the disk's edge to arrive at the location of "clock 0". V and D are positive for eastward motion of the clock. (Note that unlike the situation of Section 4.1 that used a *distance* $D > 0$, here D is a *displacement* and so can be negative.) Clock 1's reading t_1 is then compared with the reading t_0 of clock 0. We require $t_0 - t_1$.

The ECI time for clock 1 to move to the location of clock 0 is

$$\text{ECI time for trip} = \frac{\text{displacement of clock 1 on disk}}{\text{velocity of clock 1 on disk}} = \frac{D}{V}\,. \tag{4.25}$$

The times elapsed on clocks 0 and 1 during this trip are each reduced from this time D/V by each clock's gamma factor in the ECI, γ_0 and γ_1:

$$t_0 = \frac{D}{V\gamma_0}\,, \quad t_1 = \frac{D}{V\gamma_1}\,. \tag{4.26}$$

Hence

$$t_0 - t_1 = \frac{D}{V}\left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\right). \tag{4.27}$$

We know that clock 0 moves at velocity v in the ECI, in which case

$$\frac{1}{\gamma_0} = \sqrt{1 - \frac{v^2}{c^2}} \simeq 1 - \frac{v^2}{2c^2}\,, \tag{4.28}$$

where the last term results if we assume $v \ll c$, which is certainly true for Earth's rotation. If we assume small speeds, relativistic velocity addition can be approximated by non-relativistic velocity addition,[6] and so

$$\frac{1}{\gamma_1} = \sqrt{1 - \frac{(v+V)^2}{c^2}} \simeq 1 - \frac{(v+V)^2}{2c^2}\,. \tag{4.29}$$

[6] In fact, adding velocities relativistically in a rotating system is problematic and beyond the scope of this analysis. I am not aware of any discussion of it, or of more complex situations that include gravity, in the literature.

Equation (4.27) becomes

$$t_0 - t_1 = \frac{D}{V}\left(1 - \frac{v^2}{2c^2} - 1 + \frac{(v+V)^2}{2c^2}\right) = \frac{D}{c^2}\left(v + \frac{V}{2}\right). \tag{4.30}$$

Here are some examples that all use a disk that is Earth's Equatorial slice (radius 6378 km), and which spins once in a sidereal day (hence $v = 465$ m/s).

1. **Clock 1 does a full circuit east at a vanishingly small speed on the disk.** Here $V = 0$, and (4.30) becomes, using SI units throughout,

$$t_0 - t_1 = \frac{Dv}{c^2} = \frac{2\pi \times 6378 \times 10^3 \times 465}{9 \times 10^{16}} \text{ seconds} \simeq 207 \text{ ns}. \tag{4.31}$$

   That is, clock 1 that was moved east will be 207 ns behind the other clocks after a full circuit. This is because it has moved faster than the other clocks in the ECI (the inertial frame in which this relativistic analysis holds), and so has been affected more strongly by time dilation.

2. **Clock 1 does a full circuit west at a vanishingly small speed on the disk.** This is the same as item 1 above, except that D is now negative:

$$t_0 - t_1 = \frac{Dv}{c^2} = \frac{-2\pi \times 6378 \times 10^3 \times 465}{9 \times 10^{16}} \text{ seconds} \simeq -207 \text{ ns}. \tag{4.32}$$

   The westward-transported clock will end up *ahead* of the other clocks by 207 ns. This is because it has travelled slower than the other clocks in the ECI, and so has not been affected as much by time dilation.

3. **Clock 1 does a full circuit east at $V = 100$ km/h.**

$$t_0 - t_1 = \frac{D}{c^2}\left(v + \frac{V}{2}\right) = \frac{2\pi \times 6378 \times 10^3}{9 \times 10^{16}}\left(465 + \frac{100{,}000}{2 \times 3600}\right) \text{seconds} \simeq 213 \text{ ns}. \tag{4.33}$$

   [The previous version of this report had a numerical error in (4.33), which produced its erroneous result of 219 ns.]

4. **Clock 1 does a full circuit west at $V = -100$ km/h.**

$$t_0 - t_1 = \frac{D}{c^2}\left(v + \frac{V}{2}\right) = \frac{-2\pi \times 6378 \times 10^3}{9 \times 10^{16}}\left(465 - \frac{100{,}000}{2 \times 3600}\right) \text{seconds} \simeq -201 \text{ ns}. \tag{4.34}$$

   [The previous version of this report had a numerical error in (4.34), which produced its erroneous result of −195 ns.] The time differences for these last two scenarios don't have the same magnitude because time dilation is not linear in speed. (Remember that the gamma factor uses the speed in the ECI, not the speed on the disk.)

5. **Clock 1 moves 10 km east at $V = 50$ km/h.**

$$t_0 - t_1 = \frac{D}{c^2}\left(v + \frac{V}{2}\right) = \frac{10{,}000}{9 \times 10^{16}}\left(465 + \frac{50{,}000}{2 \times 3600}\right) \text{seconds} \simeq 0.05 \text{ ns}. \tag{4.35}$$

   This eastward-transported clock has thus lost 0.05 ns. This is a small amount by the needs of modern clocks, but is useful as an indicator of what can be expected in practice.

The above analysis shows the extent to which a transported clock will gain or lose time; but even so, none of it addresses the simple fact that clocks on a rotating Earth cannot be synchronised.

4.4. TAI and the Date Line

Appendix C introduces the time on Earth's geoid, known as *TAI*. Reference [19] states that a suggestion has been made that a need exists for a discontinuity in TAI to be placed at the International Date Line, although it gives no details. I argue there that any such suggestion is based on a wrong application of relativity to precise timing. (The fact that TAI incorporates gravity is irrelevant to the current discussion, since the same ideas apply to clocks fixed to the edge of a disk rotating in an inertial frame far from any gravity.)

First, picture the set of clocks shown in Figure 5. These are fixed to the edge of a disk that rotates in an inertial frame, and can be envisaged as being fixed to Earth's Equator, which rotates in the ECI. If the clocks are all synchronised in the ECI, then they will not be synchronised relative to each other. The Lorentz transform describing the MCIF at each event on the Equator produces a set of planes of simultaneity, one of which is shown in Figure 5. This plane results from applying the two-space-dimensional Lorentz transform to the MCIF of clock 0: it is the approach used in [11] and is the bread and butter of special relativity: constructing MCIFs and Lorentz transforms. In Figure 5, with all clocks synchronised in the ECI, when clock 0 displays time 0, clock 0 says that clock 90 displays 33 nanoseconds, clock 180 displays 0, and clock 270 displays −33 nanoseconds.

In contrast, the standard precise-timing approach to this set of clocks has been to apply a chain of *one*-space-dimensional Lorentz transforms around the Equator. With reference to equation (A.8): clock 0 then maintains that clock 1 reads ahead by approximately vL/c^2, where $v \simeq 465$ m/s is the clocks' speed in the ECI, L is their separation, and because $v \ll c$, we are ignoring the squared gamma factor of (A.8). Hence, clock 0 supposedly maintains that clock 1 reads ahead of clock 0 by $vL/c^2 \simeq 0.576$ ns. Similarly, clock 1 maintains that clock 2 reads ahead of clock 1 by 0.576 ns, and so on, until we get to clock 359, which maintains that clock 0 reads ahead by the same amount. The upshot of this chaining together of *one*-space-dimensional Lorentz transforms is a build-up of time differences that amounts to 360×0.576 ns $= 207$ ns back at clock 0.[7]

This approach of the precise-timing community was in fact first put forward a century ago in discussions of the rotating disk: see [1] for details. Special relativity has evolved since then to treat MCIFs with much more care. The problem with the one-space-dimensional argument above is that it chains together a set of MCIFs in a way that produces a contradiction, in that the first MCIF ends up as saying that "now" is simultaneous with a moment 207 ns into its own future. This alone should indicate that chaining MCIFs together in this one-space-dimensional way is not valid; yet, the precise-timing community persists with such a model. The use of MCIFs in relativity is far more subtle than that: they must be sewn together with care even in the simplest non-inertial frame, the uniformly accelerated frame described in Section 2.2.

The above (mis)use of MCIFs has led some in the precise-timing community to suggest that a need exists for a discontinuity in TAI to be placed at the International Date Line [19]. But such a discontinuity would be a disaster, because the very existence of that 207 ns discontinuity is based on a wrong application of relativity: namely, using a chain of one-space-dimensional Lorentz transforms when relativity calls for a single two-space-dimensional Lorentz transform, which produces no build-up of time going around the Equator. The two-space-dimensional transform simply means that such clocks cannot be synchronised in any sort of "ECEF frame".

[7] Despite the appearance of the 207 ns (well known in the Sagnac effect), this is *not* the Sagnac effect, because the Sagnac effect contains no relativity, whereas the above scenario is built on the relativity of simultaneity.

For any one observer in Figure 5, no discontinuity exists; the problem is only that different observers have different standards of simultaneity. Relativity makes no excuses for this disagreement in simultaneity, and making TAI discontinuous along some chosen meridian will not "fix" that. Nothing here needs fixing.

5. Clock Rates with Gravity Present

Earth's gravity affects clocks greatly, such that when two of these interact over large differences in altitude, their timing rates can be markedly different. Appendix A gives some background to calculating the flow of time as a function of altitude. In particular, we will use the weak-field metric (A.22) of Appendix A.2 to calculate some clock rates of interest. Use of this metric is certainly conventional, and probably quite sufficient, in the field of precise timing.

As an example of using the weak-field metric to investigate a clock's timing, suppose that we have two clocks. One is fixed to Earth's surface at sea level, and the other is fixed at the top of a tower, one kilometre vertically above the first. After one day, what is the difference in their displayed times? What about after one million years?

These two clocks are at latitude λ. The one at sea level ("height 0") counts out a time $\Delta\tau_0$. The other, at height h above this clock, counts out a time $\Delta\tau_h$. As Earth turns, for each clock r is constant. We require $\Delta\tau_h - \Delta\tau_0$ when, say, $\Delta\tau_0$ equals one million years and h equals one kilometre. Begin with

$$\Delta\tau_h - \Delta\tau_0 = \Delta\tau_0 \left(\frac{\Delta\tau_h}{\Delta\tau_0} - 1\right). \tag{5.1}$$

The weak-field metric is appropriate here. Its version in Schwarzschild coordinates, (A.21), will shortly turn out to be useful. Omitting factors of G and c (which are easily restored later), this is

$$d\tau^2 \simeq (1+2\Phi)\, dt^2 - (1-2\Phi)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 . \tag{5.2}$$

We can assume Earth is spherical; thus $\Phi = -M/r$ where M is Earth's mass. This metric gives the square of the proper time $d\tau$ between two events

$$(t, r, \theta, \phi) \quad \text{and} \quad (t + dt,\ r + dr,\ \theta + d\theta,\ \phi + d\phi) . \tag{5.3}$$

Because both r and θ are fixed for our two clocks, the square of the proper time between any two such events that a clock is present at will then be

$$d\tau^2 \simeq (1 - 2M/r)\, dt^2 - r^2 \sin^2\theta\, d\phi^2 . \tag{5.4}$$

Now apply this equation to each clock. The lower clock is at $r = R$, Earth's radius. The square of the proper time for this clock between two infinitesimally separated events is then

$$d\tau_0^2 \simeq \left(1 - \frac{2M}{R}\right) dt^2 - R^2 \sin^2\theta\, d\phi^2 . \tag{5.5}$$

The higher clock is at $r = R + h$. The square of the proper time for this clock between two infinitesimally separated events is

$$d\tau_h^2 \simeq \left(1 - \frac{2M}{R+h}\right) dt^2 - (R+h)^2 \sin^2\theta\, d\phi^2 . \tag{5.6}$$

Following the discussion just after (A.20), the Schwarzschild coordinates allow us to equate Earth's angular velocity ω in the ECI with $\mathrm{d}\phi/\mathrm{d}t$. Hence

$$\frac{\mathrm{d}\tau_h^2}{\mathrm{d}\tau_0^2} = \frac{1 - \frac{2M}{R+h} - (R+h)^2 \sin^2\theta\,\omega^2}{1 - \frac{2M}{R} - R^2 \sin^2\theta\,\omega^2}\,. \tag{5.7}$$

Now compute the relative sizes of the terms in (5.7). For this, we must restore all factors of G and c to make all the terms dimensionless, so they can be compared with the "1" in (5.7). Also refer to (C.3), and the fact that the speed of a point on Earth's Equator in the ECI is $R\omega \simeq 465$ m/s:

$$\begin{aligned} \frac{M}{R} \text{ becomes } \frac{GM}{Rc^2} &\simeq \frac{4\times10^{14}}{6.4\times10^6\times9\times10^{16}} \simeq 10^{-9}. \\ R^2\omega^2 \text{ becomes } \left(\frac{R\omega}{c}\right)^2 &\simeq \left(\frac{465}{3\times10^8}\right)^2 \simeq 10^{-12} \ll \frac{M}{R} \ll 1\,. \end{aligned} \tag{5.8}$$

It follows that we can ignore the terms involving ω in (5.7)—which also means we can ignore the latitude of the clocks. Equation (5.7) then becomes

$$\frac{\mathrm{d}\tau_h^2}{\mathrm{d}\tau_0^2} \simeq \frac{1 - \frac{2M}{R+h}}{1 - \frac{2M}{R}} \simeq \left(1 - \frac{2M}{R+h}\right)\left(1 + \frac{2M}{R}\right) \simeq 1 + \frac{2Mh}{R^2}\,. \tag{5.9}$$

Hence

$$\frac{\mathrm{d}\tau_h}{\mathrm{d}\tau_0} \simeq 1 + \frac{Mh}{R^2}\,. \tag{5.10}$$

The right-hand side of (5.10) is constant in time. That means the slope of a plot of τ_h versus τ_0 is constant in time, which allows us to say

$$\frac{\Delta\tau_h}{\Delta\tau_0} = \frac{\mathrm{d}\tau_h}{\mathrm{d}\tau_0}\,. \tag{5.11}$$

Equations (5.10) and (5.11) allow (5.1) to be written as

$$\Delta\tau_h - \Delta\tau_0 = \Delta\tau_0\left(\frac{\mathrm{d}\tau_h}{\mathrm{d}\tau_0} - 1\right) \simeq \frac{\Delta\tau_0 Mh}{R^2}\,. \tag{5.12}$$

Restoring factors of G and c gives

$$\Delta\tau_h - \Delta\tau_0 \simeq \frac{\Delta\tau_0 GMh}{c^2R^2}\,. \tag{5.13}$$

Now set $h = 1$ km and $\Delta\tau_0 = 1$ day $= 86{,}400$ s, and refer to (C.3) for the necessary parameters. Equation (5.13) becomes

$$\Delta\tau_h - \Delta\tau_0 \simeq \frac{86{,}400\times3.99\times10^{14}\times1000}{\left(3.00\times10^8\right)^2\times\left(6.38\times10^6\right)^2} \text{ seconds} \simeq 9 \text{ ns}\,. \tag{5.14}$$

Thus in one day, the upper clock counts (that is, it ages) 9 nanoseconds more than the lower clock. Should we have set $\Delta\tau_h$ to be one day rather than $\Delta\tau_0$? Doing so will not affect the answer to the accuracy we have used.

Retaining $h = 1$ km, set $\Delta\tau_0 = 1$ million years $= 31.56 \times 10^{12}$ s. Equation (5.13) becomes

$$\Delta\tau_h - \Delta\tau_0 \simeq \frac{31.56 \times 10^{12} \times 3.99 \times 10^{14} \times 1000}{\left(3.00 \times 10^8\right)^2 \times \left(6.38 \times 10^6\right)^2} \text{ seconds} \simeq 3.4 \text{ s}\,. \tag{5.15}$$

In one million years, the upper clock counts 3.4 seconds more than the lower clock.

The above analysis should not be understood as implying a notion of simultaneity that is not present in general relativity. It would be wrong for the clock at sea level to say "Simultaneous with my display showing 1 day, the clock at altitude displays 1 day plus 9 nanoseconds". We *can* say that if we could switch gravity and Earth's rotation off and zero both clocks, then switch gravity/rotation back on and let the lower clock count for one day, then switch gravity/rotation off again, the upper clock would display one day plus 9 nanoseconds. That latter statement does not use or require any concept of simultaneity when gravity and rotation are present.

Finally, the above analysis has no great effect on questions of synchronising clocks that are, for example, both at sea level. Those clocks tick at the same rate—but, as in the previous paragraph, this does *not* mean that either clock can say anything about what the other clock displays "at this moment".

5.1. Differential Flow of Time Within a Large Clock

Consider a clock whose mechanism is one metre high. At its base ($h \equiv 0$), a time $\Delta\tau_0$ of, say, one second elapses. Ten centimetres up ($h = 10$ cm), equation (5.13) says that the amount of time elapsing is

$$\Delta\tau\ (h = 10 \text{ cm}) = 1 \text{ s} + \frac{1 \text{ s} \times GM \times 0.1 \text{ m}}{c^2 R^2} \simeq \left(1 + 1.1 \times 10^{-17}\right) \text{s}. \tag{5.16}$$

The right-hand side of (5.13) is proportional to h (where h is assumed much less than R); so at the top of the clock ($h = 1$ m), the amount of time elapsing is

$$\Delta\tau\ (h = 1 \text{ m}) \simeq \left(1 + 1.1 \times 10^{-16}\right) \text{s}. \tag{5.17}$$

This variation in *space* is presumably different from the expected variation in *time* of one part in 10^{18} that is now said to characterise clocks of cutting-edge accuracy. But how these two variations might relate to each other is not clear.

Note that this gravitational variation is much larger than the 10^{-19} seconds evident in Figure 10. But the point here is that the gravitational variation can be factored in when comparing rates of clocks at different heights. (It even reduces to zero for two clocks at the same height.) In contrast, the time difference shown in Figure 10 is absolute, because it results from Earth's immutable spinning.

5.2. How Often Must Clocks be Synchronised?

Although we must abandon the idea of synchronising clocks in any known sense in the presence of gravity, the above calculation using the weak-field metric gives a notion of "correcting" the time on the clock at height h at, say, periodic intervals, to keep its time aligned with that of the clock at height zero. Consider that (5.13) gives the "excess" time T_{excess} displayed on the clock at height h:

$$T_{\text{excess}} \equiv \Delta\tau_h - \Delta\tau_0 \simeq \frac{\Delta\tau_0 GMh}{c^2R^2}\,. \tag{5.18}$$

So, we might hope to determine at least approximately the length of time $\Delta\tau_0$ that can pass on the clock at height zero before the time on the clock at height h has advanced by that same amount plus some excess T_{excess}, by inverting (5.18):

$$\Delta\tau_0 \simeq \frac{c^2R^2T_{\text{excess}}}{GMh}\,. \tag{5.19}$$

When the clocks are separated in height by $h = 100$ m and we allow their time difference to be no more than $T_{\text{excess}} = 1$ ns, (5.19) produces $\Delta\tau_0 \simeq 1$ day. That is, to maintain some form of synchronicity to within a nanosecond, we must ensure that the clocks are synchronised (however that might be done) at least once per day.

5.3. The Speed of Light and GPS

The standard algorithms used by GPS receivers employ the SI value of the speed of light: $c = 299{,}792{,}458$ m/s. How is light's speed changed by gravity, and what effect might that have on the position returned by a GPS receiver?

Assume that a GPS radio signal is sent along a radial path near Earth, in a spacetime governed by the weak-field metric (5.2), which is reasonable to use in Earth's vicinity. Setting $\mathrm{d}\theta = \mathrm{d}\phi = 0$ gives

$$\mathrm{d}\tau^2 \simeq (1+2\Phi)\,\mathrm{d}t^2 - (1-2\Phi)\,\mathrm{d}r^2\,. \tag{5.20}$$

Assume Earth is spherical, so that its potential is $\Phi = -M/r$ in units where $G = c = 1$. (The actual non-sphericity of Earth has negligible effect on the numbers below, since they concern how light's speed changes depending on whether Earth is present or not, rather than whether or not that Earth is exactly spherical.) Light travels on world lines such that $\mathrm{d}\tau = 0$, in which case (5.20) becomes

$$0 \simeq \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2. \tag{5.21}$$

With $|\Phi| \ll 1$, it follows that light's velocity is approximately

$$\frac{\mathrm{d}r}{\mathrm{d}t} \simeq \pm\left(1 - \frac{2M}{r}\right). \tag{5.22}$$

In conventional units, this is a speed of

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| \simeq c - \frac{2GM}{rc}\,. \tag{5.23}$$

Clearly this speed is less than c in Earth's vicinity, and tends toward c as the distance r from Earth goes to infinity. With $GM = 3.986 \times 10^{14}$ SI units, the discrepancy from c in (5.23) becomes

$$\text{discrepancy} = \frac{2GM}{rc} \simeq \frac{2 \times 3.986 \times 10^{14}}{\frac{r}{1\ \text{km}} \times 1000 \times 2.998 \times 10^8} \simeq \frac{2659}{r/(1\ \text{km})}\,. \tag{5.24}$$

At a distance of $r = 6400$ km from Earth's centre, this discrepancy is 42 cm/s. At a distance of $r = 26{,}000$ km from Earth's centre, the discrepancy is 0.10 cm/s. If we reduce the speed of light by these tens of centimetres per second in an algorithm that returns the position of a point on Earth based on GPS measurements, the horizontal and vertical position estimates are typically changed by some millimetres, and the estimated receiver clock error changes by about one part in 10^9. So if a typical clock was in error by one second before acquisition of a set of GPS signals, then after acquisition and the subsequent processing, it could still be in error by a nanosecond. This suggests a bound on ECI synchronisation of Earth-bound clocks via GPS (such as in two-way time transfer), that depends on the initial time error of the relevant clocks.

The Shapiro Delay

Suppose we send a beam of light radially from some r_0 to $r_1 > r_0$ in Earth's vicinity. How long does it take? Non-relativistically, we expect an answer of $(r_1 - r_0)/c$. Relativistically, light slows down near a mass, and so we expect the time to be greater than $(r_1 - r_0)/c$ by some small amount, called the *Shapiro delay*. (The existence of this delay has been verified successfully for radio signals sent across the inner Solar System.) Using (5.22), the time of flight is

$$\begin{aligned} \text{flight time} &= \int \mathrm{d}t = \int_{r_0}^{r_1} \frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r \overset{(5.22)}{=\!=} \int_{r_0}^{r_1} \frac{\mathrm{d}r}{1 - 2M/r} = \big[r + 2M\ln(r - 2M)\big]_{r_0}^{r_1} \\ &= r_1 - r_0 + 2M \ln\frac{r_1 - 2M}{r_0 - 2M}\,. \end{aligned} \tag{5.25}$$

In conventional units, this is

$$\text{flight time} = \underbrace{\frac{r_1 - r_0}{c}}_{\substack{\text{non-relativistic}\\ \text{value}}} + \underbrace{\frac{2GM}{c^3} \ln \frac{r_1 - 2GM/c^2}{r_0 - 2GM/c^2}}_{\text{Shapiro delay}}\,. \tag{5.26}$$

For Earth, with $GM \simeq 4 \times 10^{14}$ SI units,

$$\frac{2GM}{c^2} \simeq \frac{2 \times 4 \times 10^{14}}{9 \times 10^{16}}\ \text{m} \approx 1\ \text{cm}. \tag{5.27}$$

This is so much smaller than typical values of r_0 and r_1 that we can ignore it, and write

$$\text{Shapiro delay near Earth} \simeq \frac{2GM}{c^3} \ln \frac{r_1}{r_0}\,. \tag{5.28}$$

If a beam of light is sent from Earth's surface ($r_0 \simeq 6400$ km) to a low Earth-orbit satellite ($r_1 \simeq 6900$ km), the Shapiro delay is approximately

$$\frac{2 \times 4 \times 10^{14}}{27 \times 10^{24}} \ln \frac{6900}{6400}\ \text{s} \simeq 2.2\ \text{ps}. \tag{5.29}$$

That is, the signal takes 2.2 picoseconds longer to reach the satellite than we would predict without using relativity. This is negligible for the satellite: not knowing it is equivalent to making an error of half a millimeter in the satellite's estimated range. But the number is a useful indication of the size of a typical relativistic effect near Earth.

6. Concluding Comments

An important question asked in this report [see, for example, the discussion following (B.12)] is: "When does a coordinate transform equate to a bona-fide frame in relativity?" For example, suppose we start with the metric for flat spacetime in one space dimension, $\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2$, and make a Galilei transform

$$t' = t\,, \quad x' = x - vt\,. \tag{6.1}$$

Then, the metric that results,

$$\mathrm{d}\tau^2 = (1 - v^2)\,\mathrm{d}t'^2 - 2v\,\mathrm{d}t'\,\mathrm{d}x' - \mathrm{d}x'^2, \tag{6.2}$$

does not describe a true relativistic frame; whereas, if we make the usual Lorentz transform, the resulting metric $\mathrm{d}\tau^2 = \mathrm{d}t'^2 - \mathrm{d}x'^2$ *does* describe a true relativistic frame. In particular, the Lorentz transform (and not the Galilei transform) produces a time coordinate with the property that events with the same value of that time coordinate are simultaneous in the relevant frame—which is precisely what a time coordinate is meant to do. But how are we to know that, based purely on inspecting those metrics? I see this as a key question in relativity, but I am not aware of it being addressed anywhere. It cannot be waved away simply by redefining simultaneity as describing two events that have the same time coordinate, where this time coordinate has been constructed in some arbitrary way to suit the task at hand. Such a redefinition carries no real physical meaning, because then any two spacelike events can be arranged to be called simultaneous, and the concept becomes empty.[8] Indeed, placing the Galilei transform on a par with the Lorentz transform runs counter to the very existence of the field of relativity. Simultaneity runs far deeper than merely defining coordinates; it is defined by the behaviour of light in a well-known way, and is extended in a restricted fashion to non-inertial frames by using MCIFs [10].

This distinction between a coordinate change and a frame change seems currently to be absent from the discipline of relativistic precise timing. It is a change of ECI coordinates that produces a "rotating set" of ECI coordinates that are currently used in the ECEF; but these are not a fully relativistically meaningful set of ECEF coordinates; they are just the best that we currently can do. In the coming years as clocks becomes ever more precise and networks connecting them expand, I think that this lack of knowledge will create contradiction and confusion in the relativistic precise-timing community. (I think it has already: see the discussion of TAI and the Date Line in Section 4.4.) A good understanding of special relativity is required—but special relativity does not currently have a high profile in universities, where academic relativists are generally expected to devote their time to the more bankable subject of general relativity. General relativity is usually said to incorporate special relativity, but it

[8]Proof: since an observer always exists for whom two given spacelike events are simultaneous in the true sense of relativity, we would then only have to invoke the Lorentz transform appropriate to that observer to declare that the events were simultaneous, thus trivialising the whole concept of simultaneity. Note that it's certainly true that, given any two spacelike events, in principle an observer exists who says those events are simultaneous in the true sense of simultaneity; but it does not follow that for a *given* observer, a particular choice of time coordinate renders those events simultaneous.

should be said that a knowledge of general relativity does not help in analysing the famous problem of producing coordinates for the rotating disk in flat spacetime. It should also be said that the languages of general and special relativity are very different, so that familiarity with one does not imply or impart familiarity with the other. And it's worth repeating here the result of a recent survey [20] that demonstrated a poor understanding of special relativity even among academic physicists—despite those physicists rating themselves with various degrees of confidence in the correctness of the wrong answers that they provided in the survey.

The upshot is that academic relativists generally have little interest in or time to devote to special relativity, and few have an interest in precise timing. The subject of timing suffers, and becomes dominated by precision-timer analyses whose relativity content can be poor or simply wrong. For example, the subtleties of simultaneity in the context of accelerated frames, which are well known in relativistic circles and are crucial to a proper understanding of special relativity, seem to be rejected by fiat in [8]. I have pointed out other misconceptions in precise-timing literature throughout this report.

But even seemingly innocuous mistakes can be found in precise-timing literature. Consider [19], in which it is stated that observers at rest on Earth attribute the special-relativistic slowing of the tick rate of moving clocks (incorrectly equated to the Sagnac effect by the author of that reference: see Appendix B.1) to "gravitomagnetic effects—that is to say, the warping of spacetime due to spacetime terms in the general-relativistic metric tensor". It's not at all clear what "spacetime terms" means, but aside from that, such observers would still detect such a slowing of a clock's tick rate if Earth were hollow and thus spacetime were *flat*. Hence curved spacetime plays no role here: if spacetime is curved, then it is curved for all observers, and if it is flat, then it is flat for all observers. The statement appears to say, incorrectly, that spacetime curvature is observer dependent.

An example of an apparently widespread misunderstanding of the metric tensor in the precise-timing community occurs in [21]'s equation (5.49) and similar equations, which writes what it calls "the metric tensor components in the ECEF up to terms of order $1/c^2$" in a notation that is equivalent to the following metric:

$$\mathrm{d}\tau^2 = (1 + 2\Phi - v^2)\,\mathrm{d}t^2 - 2\,\mathrm{d}t\,(v_x\,\mathrm{d}x + v_y\,\mathrm{d}y + v_z\,\mathrm{d}z) - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{6.3}$$

How this metric has been produced is not made clear in [21], although it seems to be a combination of (6.2) and the weak-field metric (5.2). In particular, the space part of (6.3) has no gravitational contribution, and yet this contribution is clearly present in the weak-field metric (5.2). The point here is that we must know the relative sizes of the time and space infinitesimals in (5.2) to be able to write a "$1/c^2$ approximation" to a metric. A metric concerns *all* events in spacetime, and we can only omit terms in an integral of the proper time $\mathrm{d}\tau$ along some world line when we have applied the metric to the events *on that world line*. Such a metric as in [21]'s (5.49) is, at most, only relevant to specific set of events (ones that are connected by a slow-moving object), and yet it is incorrectly touted as an approximation to the metric of all of spacetime.

Another example of an application of relativity whose mathematics and physics are demonstrably wrong appears in [17]'s discussion of the relativity of simultaneity in an inertial frame: in a simple special-relativistic scenario whose infinitesimals (predictably) obey the Lorentz transform $\mathrm{d}x' = \gamma(\mathrm{d}x - v\,\mathrm{d}t)$, that reference says that $\mathrm{d}x'$ equals $\mathrm{d}x$; then, when the resulting expressions start to go awry, it swaps primed and unprimed coordinates to arrive at the known correct result. Indeed, if the correct expression $\mathrm{d}x' = \gamma(\mathrm{d}x - v\,\mathrm{d}t)$ is used from the outset, the correct result emerges very simply.

The success of GPS—whose relativity content is sufficient but actually very small—appears to have spawned many purportedly relativistic analyses by the precise-timing/GPS community that are assumed correct by fiat in that community, even when those analyses have no GPS content. Some of those analyses trample roughshod over subtle concepts that are still being argued about by relativity physicists a century after relativity first appeared. Up until now, what I see as a mis-handling of relativity in precise timing has caused no big problems; for example, it has no bearing on the performance of GPS. But as the world's timing requirements grow ever more stringent, I think that the chance of a poorly reasoned precise-timing analysis having adverse effects—civilian and military—is set to grow.

7. References

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[6] A reference to simultaneity whose definition is quite unclear has even appeared in the Encyclopedia Britannica's article on relativity of some years ago. There, a short paragraph describing light from two stars at unequal distances arriving at an observer appears to me to imply that the article is incorrectly defining simultaneity such that an observer who *sees* two events happen together judges them to be simultaneous. Compare this with the correct definition, which concerns when events *occur*: all times of travel of signals from those events to an observer are assumed known to the observer, who then subtracts those travel times from the current time to find the signals' times of emission.

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Appendix A. Primer on the Lorentz Transform and the Inclusion of Gravity

We discuss here a basic special-relativistic scenario that is basic to the subject. This involves two reference frames, S and S'. The S frame measures S' to move with constant velocity v in S, along the x axis of S. Cartesian coordinates for the two frames are related by the standard Lorentz transform:

$$\begin{aligned} t' &= \gamma(t - vx) + \text{constant}\,, \\ x' &= \gamma(x - vt) + \text{constant}\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{A.1}$$

where $\gamma \equiv 1/\sqrt{1-v^2} > 1$, and where

$$\{t, t'\} = c \times \text{time in } \{S, S'\}\,, \quad \text{and} \;\; v = \text{velocity}/c\,. \tag{A.2}$$

The equations in (A.1) are inverted simply by changing the sign of v, to express unprimed coordinates in terms of primed coordinates.

Both for this report and for an understanding of relativity in general, it's crucial to realise that the Lorentz transform defines coordinates for S' that are *useful* and meaningful, in that they satisfy two requirements of good coordinates:

1. events that are simultaneous in S' have the same time coordinate t', and
2. an object that doesn't move in S' has a fixed space coordinate x'.

We could certainly relate the two frames with, in fact, any arbitrary transform—even the everyday Galilei transform of non-relativistic physics, which replaces the first two lines of (A.1) with $t' = t$ and $x' = x - vt$. But such a t' and x' would not satisfy the above two requirements, and so would be very difficult to use, since the necessary relativity in any scenario would have to be injected as an extra set of procedures. In contrast, the Lorentz transform has these procedures built in.

To gain a feel for the use of the Lorentz transform, we establish the four key ingredients of special relativity.

Time dilation: This follows from the inverse of (A.1). Consider two events: two successive ticks of a clock at rest in S'. Between these two events, S records a time Δt, and S' records a time $\Delta t'$. The clock moves along the x axis with a velocity v, and we know that $\Delta x' = 0$. It follows from the inverse of (A.1) that

$$\Delta t = \gamma(\Delta t' + v\,\Delta x') = \gamma\,\Delta t'. \tag{A.3}$$

That is, $\Delta t' = \Delta t/\gamma$, meaning that a clock with velocity v ticks slowly by a factor of γ.

Length contraction: Consider two clocks, 1 and 2, at rest in S' and a distance $\Delta x'$ apart in that frame. We require their separation Δx in S. We define two events as the positions of the clocks at the *same time* in S, since such positions define the clocks' separation in S. So $\Delta t = 0$, and now consider

$$\Delta x' = \gamma(\Delta x - v\,\Delta t) = \gamma\,\Delta x\,. \tag{A.4}$$

Hence $\Delta x = \Delta x'/\gamma$: the moving clocks' separation is measured as contracted by γ.

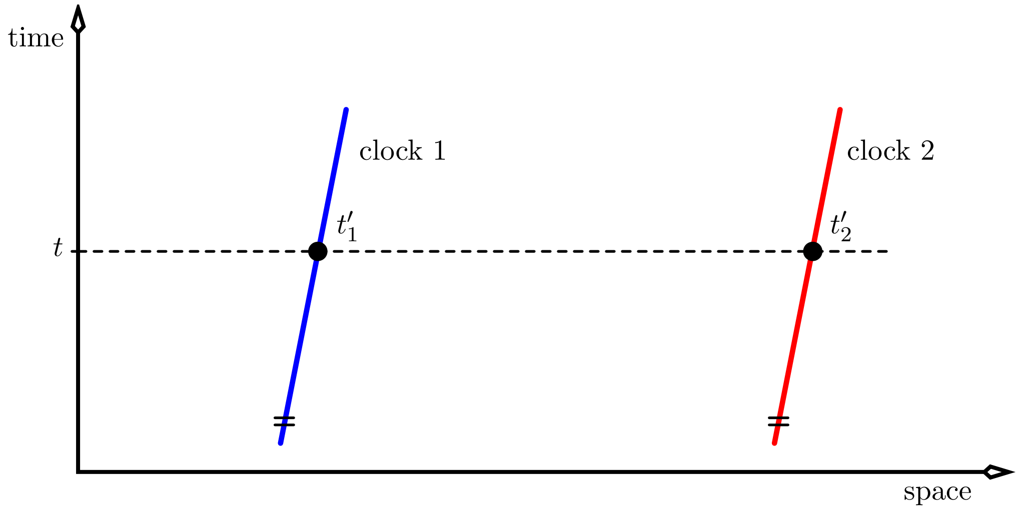

Figure 15: The times t'_1, t'_2 on clocks 1 and 2 are to be found at the same moment $t = t_1 = t_2$ in S

Different levels of synchronisation: Figure 15 shows two clocks at rest in S', that are synchronised with each other (and with all clocks at rest in S'). Compare their readings t'_1 and t'_2 at a single time $t = t_1 = t_2$ in S. Do this by applying the Lorentz transform $t = \gamma(t' + vx')$ to the equation "$t_1 = t_2$":

$$\gamma(t'_1 + vx'_1) = \gamma(t'_2 + vx'_2)\,, \tag{A.5}$$

in which case

$$t'_1 - t'_2 = v(x'_2 - x'_1) = vL'\,. \tag{A.6}$$

It follows that clock 1 displays ahead of clock 2 by the amount vL', where L' is the clocks' "rest separation". The fact that clocks synchronised in their rest frame S' are found to be unsynchronised in a frame in which they move is central to this report.

As an example, consider two clocks at rest on Earth's Equator that are relatively closely spaced compared to Earth's radius, so that they have almost the same velocity in the ECI. They can then be regarded as effectively the two clocks of equation (A.6), moving with an eastward speed of $v = 465$ m/s in the ECI. If the clocks are respectively 1 km, 100 km, and 3000 km apart and have been synchronised in the ECEF, then in the ECI we might expect that the western clock will display ahead of the eastern clock by [from (A.6), with factors of c restored]

$$\frac{vL'}{c^2} = \frac{465\ \mathrm{m/s} \times \{1\ \mathrm{km},\ 100\ \mathrm{km},\ 3000\ \mathrm{km}\}}{9 \times 10^{16}\ \mathrm{m^2/s^2}} \simeq \{5\ \mathrm{ps},\ 0.5\ \mathrm{ns},\ 15\ \mathrm{ns}\}\,. \tag{A.7}$$

The first two times on the right-hand side of (A.7) are realistic, but when the clocks' separation is comparable to Earth's radius [such as the 3000 km value in (A.7)]—they do not share the same inertial frame, and this one-space-dimensional picture starts to break down. More on this is said around equation (2.1).

Conversely, suppose that the clocks of Figure 15 have been slaved to the time of the frame in which they are moving. Then, they do not tick at their natural rate: they must be made to tick quickly by a factor of γ, since this will then cancel the slowing of

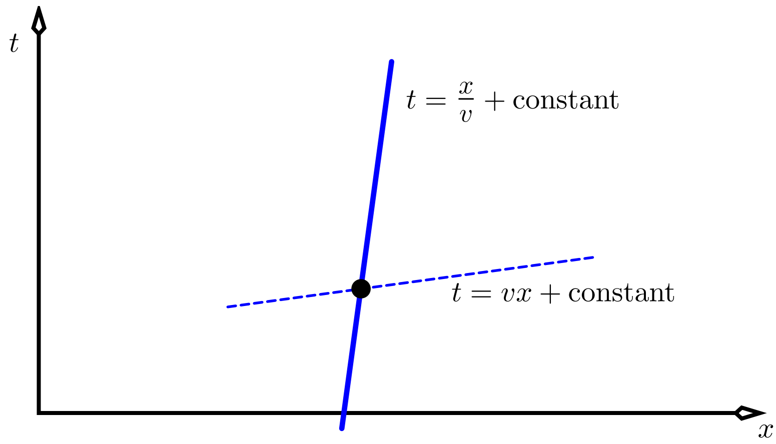

Figure 16: The dashed line shows all events that the clock following the solid world line regards as simultaneous

their tick rate by the same factor in the frame of the figure. A simple argument shows that the clocks agree that clock 2 displays a time that is later than that of clock 1 by an amount $\gamma^2 vL$, where L is the clock's separation in the frame of the figure:

$$\text{both clocks say that clock 2 leads clock 1 by } \gamma^2 vL\,. \tag{A.8}$$

We use this expression in Section 4.4.

Set of simultaneous events: Finally, we plot the set of all events that a moving clock says are simultaneous. The solid line in Figure 16 is the world line of a clock at rest in S', and thus moving at velocity v in S. We ask: what events does it say are simultaneous with the event shown as the small black disk? We require all events (t, x) such that t' for each of these events is some given number. Applying the Lorentz transform

$$t' = \gamma(t - vx) + \text{constant} \tag{A.9}$$

gives

$$\text{constant} = \gamma(t - vx) + \text{constant}. \tag{A.10}$$

It follows that the set of simultaneous events (t, x) is given by

$$t = vx + \text{constant}. \tag{A.11}$$

This set of events is shown as the red line in Figure 16. The main point here is that the line of simultaneity to a world line of slope $1/v$ has slope v. This should be borne in mind in all relevant analyses of this report. Note that if two space dimensions are being considered, the line of simultaneity becomes the *plane* of simultaneity: the set of events (t, x, y) described by (A.11) augmented with all values of y; that is, the plane in xyt space with equation $-vx + 0y + t = \text{constant}$. This has a normal vector of $(-v, 0, 1)$ in xyt coordinates. In three space dimensions, we augment this with all values of z.

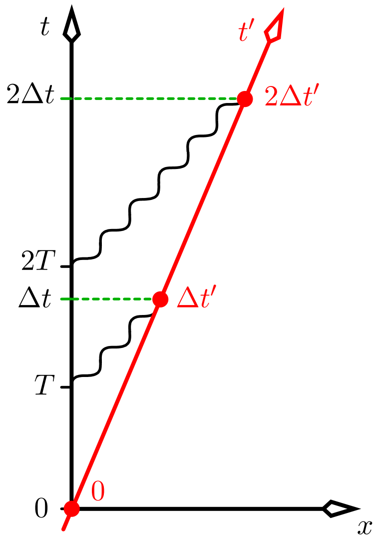

Figure 17: The case of observers who are relatively receding, with no gravity present. The frame is that of the emitter, always at $x = 0$, whose world line is the t axis. The world line of the receding "primed" receiver is red, which is then the t' axis.

A.1. Relative Motion with No Gravity

In the absence of gravity, spacetime is flat, and so can be drawn in such a way that all light rays follow straight lines on a diagram of time versus one space dimension. Figure 17 shows such a diagram where the time axis is conventionally scaled so as to make light rays (the wavy curves) run at 45° to both axes. This scaling is equivalent to calling the quantity ct "time" and then omitting the c. All factors of c will be omitted in what follows.

Consider the frame of the emitter, in which the receiver is receding at constant velocity with speed v. Suppose that initially the emitter and receiver coincide momentarily, at which moment the clocks of both their frames are set all to display zero. Special relativity tells us that the rate of flow of the emitter's time t differs from the rate of flow of the receiver's time t'.

The emitter now sends a light signal to the receiver at each of times $t = 0, T, 2T$ as displayed on the emitter's clock. The receiver receives these signals at times $t' = 0, \Delta t', 2\Delta t'$ respectively. From the emitter's perspective, those signals were received at times $t = 0, \Delta t, 2\Delta t$ respectively. Note that this last statement requires a notion of simultaneity, which is carefully defined in special relativity. The emitter says "Simultaneous with my clock displaying 0, the receiver's clock displays 0; simultaneous with my clock displaying Δt, the receiver's clock displays $\Delta t'$; and simultaneous with my clock displaying $2\Delta t$, the receiver's clock displays $2\Delta t'$." Any two events (points on the spacetime diagram) will be simultaneous for the emitter if and only if those events lie on the same horizontal line of Figure 17. In particular, portions of the infinitely long lines of simultaneity for the emitter at times $t = \Delta t$ and $2\Delta t$ are drawn as dashed green. Although all events lying on any horizontal line are regarded by the emitter as simultaneous, they are not regarded by the receiver as simultaneous.

The time intervals $\Delta t, \Delta t'$ in the ratio $\Delta t/\Delta t' = \gamma \equiv 1/\sqrt{1 - v^2}$. This follows from an analysis that incorporates the postulates of special relativity, and is not something that is supposed to be evident in the figure. Note that in particular, the ratio of rates of flow of times is *not* $T/\Delta t'$, because this latter quantity involves the travel time of light, which is not relevant to the discussion of clock rates. In fact, $T/\Delta t'$ combines the clock rates with the relevant Doppler shift.

An additional postulate of special relativity, the *Clock Postulate*, says that a clock's tick rate is not affected by its acceleration. So when the receiving clock is able to accelerate, it becomes more appropriate to consider only infinitesimal time increments, and to maintain that $\mathrm{d}t/\mathrm{d}t' = \gamma = 1/\sqrt{1-v^2}$, where γ and v are now functions of time. Consider two infinitesimally separated events, occurring at (t, x, y, z) and $(t+\mathrm{d}t, x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$. The infinitesimal time elapsed on a clock that connects them is called the "proper time" $\mathrm{d}\tau$ between those events. It follows that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}}\,. \tag{A.12}$$

That is,

$$\begin{aligned} \mathrm{d}\tau^2 = \mathrm{d}t^2(1-v^2) &= \mathrm{d}t^2\left[1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] \\ &= \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \end{aligned} \tag{A.13}$$

This expression $\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ is independent of how the clock moves, and so quantifies the geometry of spacetime itself. It is the *metric* of flat spacetime, a measure of "spacetime distance". Note that for comparison with the next section, this metric can be written in spherical polar coordinates as

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{A.14}$$

A.2. Gravity With No Relative Motion

It's natural to pursue the idea of Appendix A.1 in a context where gravity is present. The presence of gravity precludes global inertial frames, so the postulates of special relativity no longer hold. Even so, we might ask what aspects of Figure 17 might be retained. The following approach is standard in the subject [10], but is described in more detail than usual here. In particular, we make explicit use of the Clock Postulate of Appendix A.1: a necessary bridge within the logic that seems to be glossed over elsewhere.

Separate emitter and receiver by a gravity gradient and give them no relative motion. We place the emitter at some vertical position x and the receiver a height H above it. Figure 18 shows a spacetime diagram of this scenario, with the emitter again sending light signals to the receiver. Discussions of non-zero spacetime curvature don't prevent us from drawing this picture on a flat page (just as world maps are printed on flat pages), but given that the postulates of special relativity need no longer hold, we will allow for light to have a non-constant speed by giving the rays curved world lines. But we will suppose that gravity is not *changing* with time, and so the two curved world lines of light shown in the figure must be congruent. This is crucial to what follows.

As in the previous case, the emitter again sends signals "up" at emitter times $t = 0$, T, and so on, although we now need consider only these first two signals, and not the one at $t = 2T$. The receiver receives these signals at times $t' = t'_1$ and $t'_1 + \Delta t'$. Suppose we say that from the emitter's perspective, these signals were received at times $t = t_1$ and $t_1 + \Delta t$.

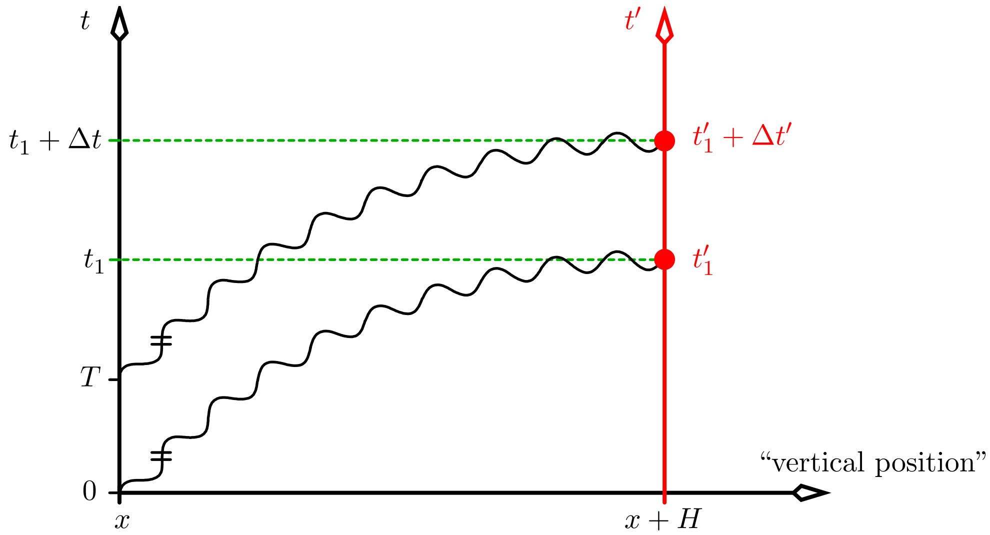

Figure 18: The case of observers who are relatively at rest in a gravity field. The frame is that of the observer at x, whose world line is the t axis. The world line of the "primed" receiver at a height H is red, which is then the t' axis.

Again, this last statement requires a notion of simultaneity. The emitter says "Simultaneous with my clock displaying t_1, the receiver's clock displays t'_1; and simultaneous with my clock displaying $t_1 + \Delta t$, the receiver's clock displays $t'_1 + \Delta t'$." At this stage, we will presume that the events defined as simultaneous by the emitter form horizontal lines on the spacetime diagram of Figure 18. Portions of the infinitely long lines of simultaneity for the emitter at times $t = t_1$ and $t_1 + \Delta t$ are drawn in dashed green.

As before, the rates of flow of times of emitter and receiver are in the ratio $\Delta t/\Delta t'$, but a new analysis is required here to calculate this ratio, since the postulates of special relativity need not (and in fact do not) apply. The correct way is to solve Einstein's equations for gravity, but we can gain some insight with the following much simpler approximate argument.

Begin by using the congruency of the light rays' world lines to infer that $T = \Delta t$. (This was not the case for special relativity in Figure 17, where T and Δt differed by the Doppler shift due to the relative motion.) We will imagine the two world lines in Figure 18 to represent successive wave fronts of a *single* light ray. The frequency of this light can now be used to give its period T, which equals Δt, which has the ratio to $\Delta t'$ (the period of the received ray) that we are seeking. We also use the idea that a light ray of frequency f has photon energy $E = hf$, where h is Planck's constant. We demand that the emitted light ray pays a gravitational tax that reduces its energy from $E = hf$ at height x to $E' = hf'$ at height $x + H$. If the gravitational field at x has newtonian potential Φ_x, then a mass m has potential energy $m\Phi_x$ there; a photon has mass E/c^2, so (omitting factors of c) the photon has potential energy $E\Phi_x$ at x. The energy that the photon gives to the gravitational field is then

$$\begin{aligned} E - E' &= \text{loss in photon "kinetic" energy} \\ &= \text{gain in photon "potential" energy} \\ &= \text{final potential energy} - \text{initial potential energy} = E\Phi_{x+H} - E\Phi_x\,. \end{aligned} \tag{A.15}$$

Putting this all together gives the ratio of clock rates as

$$\frac{\Delta t}{\Delta t'} = \frac{T}{\Delta t'} = \frac{1/f}{1/f'} = \frac{h/E}{h/E'} = \frac{E'}{E} = \frac{E - (E\Phi_{x+H} - E\Phi_x)}{E} = 1 - (\Phi_{x+H} - \Phi_x) < 1\,. \quad \text{(A.16)}$$

We see that a clock high up in a gravitational field runs fast compared to one lower down. In essence, observers at $x + H$ receive the signals from x as red-shifted because those signals lost energy in climbing up the gravitational potential. Since their clocks tell them that they are receiving successive wave fronts at a lower frequency (f') than the "factory standard" (f) and yet no Doppler shift was present (since the clocks have no relative motion), they conclude that their clocks are counting a comparatively large amount of time between successive wave fronts. Hence their clocks must be running faster than those "lower down" at x.

Suppose that we lift the upper clock infinitely far up, taking the height difference $H \to \infty$, so that $\Phi_{x+H} \to 0$ and (A.16) becomes

$$\frac{\Delta t}{\Delta t_\infty} = 1 + \Phi_x < 1\,. \quad \text{(A.17)}$$

Given that this clock at infinity is unaffected by gravity, *its* time is conventionally labelled t. Einstein's Equivalence Principle says that any clock measures the proper time between two events at its own location. Why? Because the Equivalence Principle says that the clock in gravity runs at the same rate as a clock that is accelerating upward at the appropriate rate in no gravity field; but we already know (from the Clock Postulate) that this latter clock measures the proper time between two infinitesimally separated events.

In particular, we are interested in the clock at x. So label the time on this clock as τ, and write (A.17) as

$$\Delta\tau = (1 + \Phi)\,\Delta t\,. \quad \text{(A.18)}$$

For the space outside a spherically symmetric mass M, the gravitational potential is a function only of the distance r from the centre of the mass, and is $\Phi(r) = -GM/r$, omitting a factor of c^2 as is conventional. Equation (A.18) becomes

$$\Delta\tau = (1 - GM/r)\,\Delta t\,. \quad \text{(A.19)}$$

Comparing this with the gravity-free case (A.13) or (A.14), we see no dependence in (A.19) on increments of space variables. This is because the two clocks in Figure 18 were at rest. Other analyses can be made that do incorporate space increments: see, for example, Section 12.3.1 of [3]. A full analysis requires Einstein's equation of general relativity, which is a postulate for the subject. For a spacetime whose geometric structure is assumed to be spherically symmetric, this equation yields the *Schwarzschild metric* outside a spherically symmetric mass:

$$\mathrm{d}\tau^2 = (1 - 2GM/r)\,\mathrm{d}t^2 - (1 - 2GM/r)^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\,, \quad \text{(A.20)}$$

using "Schwarzschild coordinates", where the radial coordinate r no longer precisely has the meaning of radial distance. In these coordinates, the circumference of a circle of radius r is $2\pi r$. More generally, when the mass M is only approximately spherically symmetric, the *weak-field* solution of Einstein's equations turns out to be, using Schwarzschild coordinates,

$$\mathrm{d}\tau^2 \simeq (1 + 2\Phi)\,\mathrm{d}t^2 - (1 - 2\Phi)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\,, \quad |\Phi| \ll 1\,. \quad \text{(A.21)}$$

In "isotropic polar coordinates", the weak-field metric is

$$d\tau^2 \simeq (1 + 2\Phi)\, dt^2 - (1 - 2\Phi) \left[dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \quad |\Phi| \ll 1\,. \tag{A.22}$$

(Isotropic refers to the fact that the speed of light is independent of direction when using these coordinates.) As expected, these metrics reduce to (A.14) in the limit $\Phi \to 0$. Equivalently, consider the case of $M = 0$ with a clock moving in a circle in no gravity at constant r with $\theta = 90°$. The Schwarzschild metric (A.20) reduces to

$$d\tau^2 = dt^2 - r^2\, d\phi^2 = dt^2 \left[1 - r^2\, (d\phi/dt)^2\right]. \tag{A.23}$$

But $r\, d\phi/dt$ is just the velocity v of the clock, so (A.23) becomes

$$d\tau = dt\, \sqrt{1 - v^2}\,, \tag{A.24}$$

which matches (A.12). This demonstrates how special-relativistic time dilation is present in general relativity.

For the case of constant r, θ, ϕ and the weak-field limit $|\Phi| \ll 1$, (A.21) and (A.22) both yield approximately (A.18) [and (A.20) yields (A.19)], but not exactly. The reason for the caveat "not exactly" is perhaps that this analysis uses the newtonian idea of potential; but it could also be that an approximation is present in the analysis of Figure 18. What can certainly be questioned in that figure is the idea of simultaneity: drawing straight lines to denote events that can be regarded as simultaneous with each other, and thus all occurring at the indicated times t_1 and $t_1 + \Delta t$. A clear definition of simultaneity is actually not present in general relativity. If I am to synchronise my clock with a clock elsewhere in a gravity field, to be able to say "At my time 12:00, you must set your clock to display 12:00" demands a concept of simultaneity that is absent from the theory. For more analysis of these metrics, see Appendix C.

The above analysis of Figure 18 required the gravitational field to have no time dependence. If the field did have such a dependence, the two world lines of light in the figure would no longer necessarily be congruent. We would then no longer know how T relates to Δt. We know T, the period of the emitted light; but we require Δt, to compare it with $\Delta t'$, the period of the received light, and we were only able to set $\Delta t = T$ in Figure 18 by using the symmetry arising from the unchanging nature of successive light rays, because the conditions of the scenario are static. In contrast, a real scenario's gravity might change with time. At this point, it's wise to resort to the full form of Einstein's equation of general relativity. Nonetheless, such analyses are not widespread in the subject, and these very fundamental questions of simultaneity and the flow of time in a changing gravity field remain ponderous in relativity.

Because Einstein's equations are too difficult to solve exactly for a mass distribution as complicated as Earth's, the weak-field metric (A.22) forms the basis of all current timing analyses. The potential Φ is often written for the case of an oblate spheroid Earth, usually as the approximation in (C.2) of Appendix C. But given that the weak-field metric (A.22) is valid only to first order in Φ, adding small refinements to it will be meaningless when those refinements are of second order in Φ, as discussed briefly in Appendix C.2. Many refinements to Φ can be found in publications of the International Astronomical Union, but I have not checked the relative sizes of the various terms.

As regards the quantitative use of high-accuracy expressions for Φ, a complicating factor is that many in the geodesy and precise-timing communities, along with the International

Astronomical Union (IAU), redefine the gravity potential to be -1 times the physicist's gravity potential. So, where a physicist writes $\Phi = -GM/r$, the IAU writes "$\Phi = +GM/r$". The IAU then compensates for this redefinition by changing the sign of Φ in the weak-field metric (A.21), and no doubt in many other equations as well. In particular, when the IAU adds refinements to the potential by writing "$\Phi = GM/r\,(1+\varepsilon)$" where ε is small, we must use great care to decide whether this expression should be corrected to become $\Phi = -GM/r\,(1+\varepsilon)$, or $\Phi = GM/r\,(-1+\varepsilon)$. I suspect that both conventions exist in the literature, with no information being given on which is being used. (I presume that geodesists, precise-timers, and the IAU *don't* redefine the *electrostatic* potential to be -1 times the physicist's electrostatic potential. Whereas the physicist's potential is the product of much logical thought that gives it a single definition regardless of that potential's source, the IAU's definition is presumably source dependent. That can only lead to the unnecessary difficulty highlighted above.)

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Appendix B. Alternative Derivation and Comments on the Sagnac Effect

Here is an alternative derivation of T_1 and T_2 in (4.5) and (4.8) that gives insight into a related area of relativity: the question of a bona-fide change of frame versus a trivial change of coordinates. We set $c = 1$ for the start of this discussion.

Taking our cue from the ECI picture in Figure 12, consider the metric for the assumed flat spacetime of the ECI in polar coordinates. We are dealing with a disk: a two-space-dimensional problem, so will use polar coordinates r, ϕ; we use ϕ rather than the conventional θ because θ already appears in Figure 12:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\phi^2. \tag{B.1}$$

The simplest Sagnac scenario that we are analysing has $r = R$, the disk radius: we are concerned only with events on the disk rim. Hence

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - R^2\,\mathrm{d}\phi^2. \tag{B.2}$$

For light, $\mathrm{d}\tau = 0$, and so

$$\mathrm{d}t^2 = R^2\,\mathrm{d}\phi^2\,. \tag{B.3}$$

If the light travels "eastward" around the disk, then $\mathrm{d}\phi/dt > 0$, and (B.3) becomes

$$\mathrm{d}t = R\,\mathrm{d}\phi\,. \tag{B.4}$$

The ECI time taken for its trip—T_1 in Figure 12—is $T_1 = \int \mathrm{d}t$.

Now consider Figure 12 in a somewhat *galileian* way: as the light pulse traverses the disk edge, in a time dt it gets "carried along" with the disk (which has angular velocity ω in the ECI); plus, it traverses a little bit $\mathrm{d}\theta$ of the total angle θ; hence

$$\mathrm{d}\phi = \omega\,\mathrm{d}t + \mathrm{d}\theta\,. \tag{B.5}$$

We emphasise that this picture is galileian: it assumes the light is carried along with the disk at the speed of the disk, which we know is not really true; after all, a light ray traversing a stream of water is *not* carried along with the speed of the water. Relativistically, if light traverses a medium of refractive index n that itself is travelling with velocity v in an inertial frame, then the speed of light in the inertial frame will be

$$\text{speed of light in inertial frame} = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{B.6}$$

(This constitutes the famous Fizeau experiment.) That is, the galileian statement of "speed of light in inertial frame equals speed of light in medium (c/n) plus speed of medium in inertial frame (v)" is only true in the limit $n \to \infty$. Equations (B.4) and (B.5) combine to give

$$\mathrm{d}t = R\omega\,\mathrm{d}t + R\,\mathrm{d}\theta\,. \tag{B.7}$$

It follows that

$$\mathrm{d}t = \frac{R\,\mathrm{d}\theta}{1 - R\omega}\,. \tag{B.8}$$

For a small disk edge speed $R\omega \ll 1$, this approximates to

$$\mathrm{d}t \simeq R\,\mathrm{d}\theta\,(1 + R\omega) = R\,\mathrm{d}\theta + R^2\omega\,\mathrm{d}\theta\,. \tag{B.9}$$

The time for the light pulse to traverse angle θ on the disk is then

$$T_1 = \int_{\theta=0}^{\theta} \mathrm{d}t \simeq R\theta + R^2\omega\theta\,. \tag{B.10}$$

T_2 can be calculated from the same procedure by imagining the disk to be spinning the other way. The result of that is the simple substitution $\omega \to -\omega$ in (B.10):

$$T_2 \simeq R\theta - R^2\omega\theta\,. \tag{B.11}$$

Finally, reinsert c, write $v \equiv R\omega$, and $D \equiv R\theta$:

$$\left\{ \begin{matrix} T_1 \\ T_2 \end{matrix} \right\} \simeq \frac{D}{c} \pm \frac{Dv}{c^2}\,. \tag{B.12}$$

These match (4.5) and (4.8) in the limit of small rotational speed. This limit of slow rotation—which gave rise to the galileian approximation used above—is a key point here. Namely, despite the use of a metric in (B.1) and the look of (B.5) (or more specifically, its non-infinitesimal version $\phi = \omega t + \theta$), the above procedure was *not* a transform to a "rotating frame". Instead, (B.5) created the angular coordinate θ of a set of "rotating coordinates" in the *inertial* ECI, $\{t, r, \theta\}$. The bottom line is that rotating ECI coordinates, which are used in our modern world in the ECEF, are not ECEF coordinates: see the comments at the start of Section 4.

The above procedure appears in a more convoluted form in Section 6 of [17], where it *is* described as a transform to a rotating frame (the ECEF), supposedly with relativity built in, constructed on the non-infinitesimal version of (B.5) (written in [17] as $\phi = \phi' + \omega_E t'$ with $t' = t$). But we see here that it contains no relativity; it is simply a galileian transform written in relativistic language. For further discussion of this, see the start of Section 6.

B.1. Transporting a Clock is Not the Sagnac Effect

The relativistic slowing of the tick rate of a clock moving on Earth, discussed in Section 4.3, is sometimes called the Sagnac effect in timing literature, such as in [17]. This is probably because the Dv/c^2 that appeared in (4.31) also appeared in (B.12). Nonetheless, the above procedure of transporting a clock has nothing to do with the Sagnac effect. The reason is that the moving-clock equation, (4.31), compares the different *relativistic* slowings of time on a set of clocks (it is calculating and comparing the different "gamma-factor" slowings of the clocks as they move in the ECI), whereas the Sagnac effect concerns a light beam chasing a moving receiver, or a race between two light beams, and requires *no* relativity for its main analysis. (It *is* modified by the relativistic slowing of the clock on a moving receiver, but this is not the main content of the Sagnac effect.)

To see this more fully, consider the Sagnac calculation of Section 4.1. A main quantity of interest is the difference between the actual transit time, say T_1 in (4.4), and the value D/c that we would expect if light travelled everywhere at c from the viewpoint of the disk. We might think that to be fully relativistic, we must include a gamma-factor $\gamma = c/\sqrt{c^2 - v^2}$ for the slowing of the clock's tick rate in the inertial frame in which the disk spins. But in fact that's not the case. The rotating clocks have different standards of simultaneity than that of the inertial frame: if clocks 1 and 2 in Figure 12 are synchronised in the inertial frame, then they are not synchronised from the viewpoint of the clocks, and so neither can simply divide the times T_1 and T_2 by γ. So as before, we insist that $v \ll c$. Then, the difference between T_1 and D/c is

$$\text{Sagnac quantity of interest} = T_1 - \frac{D}{c} = \frac{D}{c-v} - \frac{D}{c} = \frac{Dv}{c(c-v)}\,. \tag{B.13}$$

To first order in v, this quantity is Dv/c^2, as written in (B.12). Now consider moving a clock, where we allow an arbitrary v but still take the limit $V \to 0$. The quantity of interest here is, from the discussion at the start of this section,

$$\begin{aligned}\text{moving-clock quantity of interest} &= \lim_{V\to 0} t_0 - t_1 \overset{(4.27)}{=\!=} \lim_{V\to 0} \frac{D}{V}\left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\right)\\ &= \lim_{V\to 0} \frac{D}{V}\left(\sqrt{1-\frac{v^2}{c^2}} - \sqrt{1-\frac{(v+V)^2}{c^2}}\right) = \frac{Dv}{c\sqrt{c^2-v^2}}\,,\end{aligned} \tag{B.14}$$

where the last result follows from L'Hôpital's rule. In the limit $v \ll c$, this quantity also becomes Dv/c^2. But clearly (B.13) and (B.14) are completely different results, despite having the same low-v limit. That they share the same limit is not surprising, because the expression "distance × velocity" occurs frequently in special relativity. Hence, moving a clock is not an instance of the Sagnac effect. A similar appearance of "Sagnac" in the context of a moving clock is in [22], equations (5) and (6). There, what amounts to the $(v + V)^2$ in the gamma factor of our (4.29) is expanded as $v^2 + V^2 + 2vV$. The cross term analogous to $2vV$ is then seemingly arbitrarily called the Sagnac effect in that paper, despite having no relation to Sagnac's non-relativistically valid scenario of light chasing a clock.

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Appendix C. Definitions of Various Times

In this appendix we discuss Earth's *geoid*. The geoid is a mean sea level over the planet: it is an equipotential surface, where the equipotential includes a centrifugal term for observers at rest on the geoid, who thus rotate with Earth.

Because Earth's geoid is an equipotential surface, no work is done on or by a photon that connects two events on the geoid, and it follows that the photon's frequency is unchanged from emission to reception. From the discussion of Appendix A.2, we infer that all clocks at rest on the geoid (thus rotating with Earth) tick at the same rate, because with no Doppler shift to complicate what is *seen* when one clock on the geoid views another (since they are relatively at rest), what is *seen* reflects reality.[9] This common tick rate of clocks at rest on the geoid is called *International Atomic Time* (TAI), denoted t_{TAI} here. It is what accurate laboratory clocks on Earth measure: its base unit is the SI second.

To examine TAI in detail, we begin with the weak-field metric in Schwarzschild spherical polar coordinates r, θ, ϕ that is derived in textbooks on general relativity and mentioned previously in (A.21):

$$\mathrm{d}\tau^2 \simeq (1+2\Phi)\,\mathrm{d}t^2 - (1-2\Phi)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2\,, \quad |\Phi| \ll 1\,. \tag{C.1}$$

Here $\mathrm{d}\tau$ is the proper time between any two infinitesimally spaced events at (t, r, θ, ϕ); Φ is the dimensionless gravitational potential at the point (r, θ, ϕ) (that is, Φ is the ECI gravitational potential divided by c^2); and in these coordinates, a circle of radius r has circumference $2\pi r$—which is a desirable feature when discussing our spheroidal Earth. With this metric, what proper time elapses on a clock at a fixed position at spatial infinity ($r \to \infty$)? There the potential tends toward zero, and (C.1) becomes $\mathrm{d}\tau^2 = \mathrm{d}t^2$. The coordinate t is thus the time on a motionless clock at spatial infinity.

To some degree of precision (discussed in Appendix D), the weak-field metric describes spacetime in the ECI. To the precision required here, Earth's potential is (in dimensionless form)

$$\Phi = \frac{-GM}{rc^2}\left[1 - \frac{a^2 J_2}{2r^2}(3\cos^2\theta - 1)\right], \tag{C.2}$$

where the following parameters appear:

$$\begin{aligned}
GM &\equiv \text{gravitational constant} \times \text{Earth's mass} \simeq 3.9860\times 10^{14}\ \mathrm{m^3/s^2},\\
r &\equiv \text{distance from Earth's centre},\\
c &\equiv \text{speed of light} \simeq 2.998\times 10^{8}\ \mathrm{m/s},\\
a &\equiv \text{Earth's WGS-84 equatorial radius} \simeq 6{,}378{,}137\ \mathrm{m},\\
J_2 &= \text{largest ``zonal harmonic coefficient''} \simeq 0.00108\,,\\
\theta &\equiv \textit{geocentric}\ \text{co-latitude (not}\ \textit{geodetic}\ \text{co-latitude).}
\end{aligned} \tag{C.3}$$

We will also need:

$$b \equiv \text{Earth's WGS-84 polar radius} \simeq 6.3568\times 10^{6}\ \mathrm{m}. \tag{C.4}$$

[9] These statements don't depend on the shape of the geoid, and hence disprove the claim in [19] that the independence of tick rate on position arises from the fine details of Earth's non-spherical mass distribution.

Now return to the metric (C.1) where Earth's centre is at $r = 0$, and consider a clock at rest on the geoid. Its $\mathrm{d}r$ and $\mathrm{d}\theta$ are both zero as time passes. The proper time squared between two infinitesimally spaced events on this clock is then, from (C.1),

$$\mathrm{d}t_{\mathrm{TAI}}^2 = (1 + 2\Phi)\,\mathrm{d}t^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 . \tag{C.5}$$

Since all clocks on the geoid tick at the same rate, we can say that

$$\frac{\text{time elapsed on clock}}{\text{time elapsed at spatial infinity}} = \frac{\mathrm{d}t_{\mathrm{TAI}}}{\mathrm{d}t} = \text{a constant at all points on the geoid.} \tag{C.6}$$

But (C.5) says that

$$\frac{\mathrm{d}t_{\mathrm{TAI}}^2}{\mathrm{d}t^2} = 1 + 2\Phi - r^2 \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 . \tag{C.7}$$

We infer that the right-hand side of (C.7) is a constant at all points on the geoid: call it $1 + 2A$:

$$\frac{\mathrm{d}t_{\mathrm{TAI}}^2}{\mathrm{d}t^2} = 1 + 2A\,. \tag{C.8}$$

Now write Earth's spin rate in the ECI—that is, with respect to the distant stars—as (remembering that t is really "$c \times$ time")

$$\omega \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t} \simeq \frac{2\pi}{c \times 86{,}164\ \mathrm{s}} \simeq 2.4324 \times 10^{-13}\ \mathrm{m}^{-1}, \tag{C.9}$$

since Earth turns once in a sidereal day of 23 hours, 56 minutes, 4 seconds (86,164 seconds). Combine (C.7) with (C.8) to infer that at all points on the geoid,

$$A = \Phi - 1/2\, r^2 \omega^2 \sin^2\theta = \text{constant}\,. \tag{C.10}$$

How do the values of the quantities in (C.10) compare? On Earth's Equator (where $\sin\theta$ is largest):

$$\Phi \simeq \frac{-GM}{ac^2} \simeq \frac{-3.9860 \times 10^{14}}{6{,}378{,}137 \times 9 \times 10^{16}} \simeq 7 \times 10^{-10},$$
$$r^2\omega^2 = a^2\omega^2 \simeq \left(6{,}378{,}137 \times 2.4324 \times 10^{-13}\right)^2 \simeq 2 \times 10^{-12}. \tag{C.11}$$

Clearly, the gravity term is dominant here.

It so happens that the expression $\Phi - 1/2\, r^2\omega^2 \sin^2\theta$ appears in non-relativistic classical mechanics as the *effective potential* Φ_{eff} on a rotating Earth. When an effective potential Φ_{eff} is defined by its effect on a test mass m in a rotating frame [that is, $-\nabla(m\Phi_{\mathrm{eff}})$ is defined to be the sum of gravity and centrifugal force acting on m], then Φ_{eff} turns out to equal $\Phi - 1/2\, r^2\omega^2 \sin^2\theta$. So we conclude that $A = \Phi_{\mathrm{eff}}$. Equation (C.8) is then usually written as

$$\frac{\mathrm{d}t_{\mathrm{TAI}}^2}{\mathrm{d}t^2} = 1 + 2\Phi_{\mathrm{eff}}\,, \tag{C.12}$$

with

$$\Phi_{\mathrm{eff}} = \Phi - 1/2\, r^2\omega^2 \sin^2\theta = \text{constant on geoid}\,. \tag{C.13}$$

We can calculate the value of this constant in (C.13) by evaluating it, say, at the Equator, where $\theta = 90°$. Equation (C.13) becomes

$$
\begin{aligned}
\Phi_{\text{eff}} &\overset{(C.13)}{=\!=} \Phi_{\theta=90°} - {}^1\!/\!_2\, a^2\omega^2 \overset{(C.2)}{=\!=} \frac{-GM}{ac^2}\left[1 - \frac{\cancel{a^2} J_2}{2\cancel{a^2}} \times -1\right] - \frac{a^2\omega^2}{2}, \\
&\simeq \frac{-3.9860 \times 10^{14}}{6{,}378{,}137 \times (2.998 \times 10^8)^2}\left[1 + \frac{0.00108}{2}\right] - \frac{1}{2}\left(6{,}378{,}137 \times 2.4324 \times 10^{-13}\right)^2 \\
&\simeq -6.9693 \times 10^{-10}. \qquad \text{(C.14)}
\end{aligned}
$$

The (positive) constant 6.9693×10^{-10} is often called L_G in the literature.[10]

Compared to a clock at spatial infinity, the rate of flow of time on the geoid is, from (C.12) and (C.14),

$$\mathrm{d}t_{\text{TAI}}/\mathrm{d}t = 1 + \Phi_{\text{eff}} \simeq 1 - L_G\,. \qquad \text{(C.15)}$$

That is, clocks on the geoid tick slightly slower than stationary clocks infinitely far from Earth. Nominally, the GPS system uses TAI as its time, but for historical reasons, GPS and TAI differ by 19 seconds:

$$t_{\text{GPS}} = t_{\text{TAI}} - 19 \text{ seconds}. \qquad \text{(C.16)}$$

TAI has the same rate as a historical relic, *terrestrial time* (TT), which has been used to set the start point of TAI:

$$t_{\text{TAI}} = t_{\text{TT}} - 32.184 \text{ seconds exactly}. \qquad \text{(C.17)}$$

The time t on a stationary clock at spatial infinity is called *geocentric coordinate time* (TCG), and is the time coordinate used in the ECI in Earth's vicinity (but not on board GPS satellites). Equations (C.15)–(C.17) then say

$$\Delta t_{\text{TAI}} = \Delta t_{\text{GPS}} = \Delta t_{\text{TT}} = (1 - L_G)\,\Delta t_{\text{TCG}}\,. \qquad \text{(C.18)}$$

At distances farther from Earth, TCG is replaced by another time coordinate, *barycentric coordinate time* (TCB). Barycentric coordinate time attempts to include the gravity fields of all Solar System objects in its metric. This metric appears in a standard document of the IAU [23], but details of how it was produced appear to be absent from [23].

C.1. The Distinction Between TAI and UTC

TAI is a uniform time that uses the SI second as its basic unit. It is effectively a reference clock that ticks in a predictable way, and is never stopped. (That is, we'll see shortly that it has no leap seconds.)

If Earth's orbit were exactly circular and Earth had no tilt, then by definition, one *solar day* of fixed length would elapse between the Sun being at the same place in the sky on two consecutive days: say, from midday (when the Sun was at the meridian) to midday. Because Earth's orbit is not exactly circular and Earth is tilted, one *mean solar day* is defined with

[10] Note that we could also evaluate Φ_{eff} at one of the poles to arrive at the same number. Since that pole calculation makes use of Earth's WGS-84 polar radius $b \simeq 6.3568 \times 10^6$ m, combining that pole calculation with (C.14) allows J_2 to be expressed in terms of the two radii a and b.

reference to Earth's rotation in a mean sense relative to the Sun (or relative to a "mean Sun"), and is defined to be exactly $24 \times 3600 = 86{,}400$ *mean solar seconds* long.

But the SI definition of a second is such that a mean solar day currently lasts about 86,400.001 SI seconds. We require our clocks to count SI seconds—but we also require them to display 00:00:00 at each midnight. These two requirements conflict. Specifically, imagine a clock that displays the TAI time of Appendix C, a "TAI clock". This displays "0 days, 00:00:00" at midnight at the start of the year. After one mean solar day, this clock displays 86,400.001 (SI) seconds, or "1 day, 00:00:00.001". So, the TAI clock is reading 0.001 seconds ahead of what we want our civilian clocks to display: "1 day, 00:00:00". After two mean solar days, the TAI clock reads $2 \times 86,400.001$ (SI) seconds, or "2 days, 00:00:00.002".

The TAI clock is, in a sense, running too quickly for civilian purposes. 365 mean solar days later, it displays $365 \times 86,400.001$ (SI) seconds, or "365 days, 00:00:00.365". After three years, it is displaying fully one second ahead of desired civilian time. We would like to stop it for one second, but if we did that, it would no longer be a TAI clock. Instead, we make a copy of the TAI clock, and stop that copy for one second; then we call this new clock a "UTC clock", where UTC stands for *coordinated universal time.* When the UTC clock ticks, it counts SI seconds, meaning it ticks at the same rate as a TAI clock. But every few years, we stop the UTC clock for one second to bring it back into line with the civilian requirement of where the Sun should be at a given time. The time on the UTC clock is *UTC time.* Both TAI and UTC use the SI second, but whereas the TAI clock never stops, the UTC clock is sometimes stopped, and the second for which it is stopped is called a *leap second.*

The mean solar day is represented by a "mean Sun" that rises in the east and sets in the west. Because TAI clocks effectively run too fast for the Sun's motion, it's as if the mean Sun moves too slowly in the sky, and occasionally we must stop our clocks for an agreed time (the leap second) to let it catch up, so that it will once again be due north at midday.

Note that the commonly found statement "Leap seconds are needed to account for Earth's rotation gradually slowing down" is actually wrong: although Earth's rotation is slowing, it is certainly not slowing so quickly that our clocks have a one-second mismatch with the Sun's position every three years. (It would have stopped spinning long ago if that were the case.) Even if Earth's spin rate stopped slowing completely, we would still have to insert leap seconds into UTC time. Clocks that tick once per SI second simply outpace the Sun, and so occasionally we must pause them for a second to allow the Sun to "catch up". This pausing of clocks concerns Earth's current angular *velocity*, and it would still be needed if Earth's spin rate stopped slowing, since the latter concerns Earth's angular *acceleration.* Even if Earth's spin rate began to *increase*, we would still need to pause our UTC clocks for the occasional second, for perhaps some years until the Sun was moving quickly enough in the sky to match the speed of an SI clock. If Earth's spin rate increased to the point that the Sun was moving "too quickly"in the sky, then we would occasionally have to do the opposite of pausing the UTC clock for a second: every few years, we would need to make a specified minute have only 59 seconds instead of the 61 seconds that occurs at present when a leap second is required.

UTC time is the modern evolved form of Greenwich Mean Time. UTC is perhaps defined more rigorously than GMT ever was, and so the two terms are now synonymous.

C.2. The Role of J_2 in Φ

Because Φ appears in the weak-field metric (C.1) to first order only, it's worthwhile to establish whether the absence of terms of higher order in Φ is potentially of greater consequence than including J_2 in the expression for Φ in (C.2), which models Earth's slight oblateness. We need only compare Φ for a spherical Earth (say, with Earth's equatorial radius a), Φ for our oblate Earth at Equator and a pole, and Φ^2 for a spherical Earth.

Spherical Earth:

$$\Phi_{\text{sphere}} = \frac{-GM}{ac^2} \simeq -6.9535 \times 10^{-10}. \tag{C.19}$$

Oblate Earth: Use (C.2):

$$\begin{aligned}
\Phi_{\text{Equator}} &\simeq \frac{-GM}{ac^2}\left[1 + \frac{J_2}{2}\right] \simeq -6.9573 \times 10^{-10}, \\
\Phi_{\text{Pole}} &\simeq \frac{-GM}{bc^2}\left[1 - \frac{a^2 J_2}{b^2}\right] \simeq -6.9693 \times 10^{-10}.
\end{aligned} \tag{C.20}$$

Second-order term: We don't know what the coefficient of Φ^2 is in a more exact version of the weak-field metric (C.1), so will assume it to be of order one. In that case, we need examine only Φ^2, and it suffices to calculate it for a sphere:

$$\Phi^2_{\text{sphere}} \simeq \left(-7 \times 10^{-10}\right)^2 \simeq 5 \times 10^{-19}. \tag{C.21}$$

The difference between the potentials for a sphere (C.19) and an oblate spheroid (C.20) is around 10^{-12}, which is about a million times larger than Φ^2. So it's reasonable to include J_2 in the expression for the potential Φ in the weak-field metric (C.1), while excluding terms that are higher order in Φ.

C.3. The ECI Speed of Light Near Earth's Surface

Section 4.2 requires the speed of light in the ECI near Earth's surface. We calculate that speed in this section by using the fact that spacetime near Earth's surface can be described using the weak-field metric (C.1). The light rays in the simple model of Section 4.2 travel along circles of fixed latitude. Using TAI time and Schwarzschild polar coordinates, their speed is

$$c' = r \sin\theta \left|\frac{\mathrm{d}\phi}{\mathrm{d}t_{\text{TAI}}}\right|. \tag{C.22}$$

Equation (C.15) says that $\mathrm{d}t_{\text{TAI}} = (1 + \Phi_{\text{eff}})\,\mathrm{d}t$. Light rays connect events separated by zero proper time. Setting $\mathrm{d}\tau$ to zero in (C.1) on a circle of fixed latitude then gives

$$(1 + 2\Phi)\,\mathrm{d}t^2 = r^2 \sin^2\theta\,\mathrm{d}\phi^2\,, \tag{C.23}$$

and hence $r \sin\theta\,|\mathrm{d}\phi/\mathrm{d}t| \simeq 1 + \Phi$. It follows that

$$\begin{aligned}
c' &= r \sin\theta \left|\frac{\mathrm{d}\phi}{\mathrm{d}t}\right| (1 - \Phi_{\text{eff}}) \simeq (1 + \Phi)(1 - \Phi_{\text{eff}}) \\
&\simeq 1 + \Phi - \Phi_{\text{eff}} \overset{\text{(C.13)}}{=\!=} 1 + {}^1\!/_2\, r^2 \omega^2 \sin^2\theta = 1 + v^2/2\,,
\end{aligned} \tag{C.24}$$

where v is the speed of the Earth-fixed clock in the ECI. In conventional units, this result "$c' = 1 + v^2/2$" becomes $c' = c + v^2/(2c)$. Because we have tended to disregard factors of v^2/c^2 throughout this report, it's sufficient to write $c' = c$, so that light on Earth's surface can be modelled as having speed c in the ECI, to first order in Earth's spin rate. This speed is used in Section 4.2.

It might be said that when TAI time is used, light travels slightly faster than c over Earth's surface because time flows slightly slowly near Earth's surface: TAI clocks tick slightly slower than clocks at infinity, as stated immediately following (C.15).

Appendix D. Limitation of the Weak-Field Metric

A set of clocks to be synchronised might be fixed to the surface of the rotating Earth. In Newton's theory, the gravity field of a rotationally symmetric body is unaffected by whether the body rotates. Up until now, following convention, we have employed this non-relativistic idea by modelling an Earth-fixed clock's timing as that of a clock *moving* in the gravity field of a non-rotating Earth. Such a field is described up to standard levels of accuracy equally well by the weak-field and Schwarzschild metrics. These metrics have been tested experimentally to some level of accuracy in the famous Hafele–Keating experiment of the early 1970s, in which the timing of clocks flown around the world for some days was measured and found to agree with this non-rotating-Earth model to the level of about one standard error, which was about 10–20 ns.

But, strictly speaking, the weak field and Schwarzschild metrics are not those of a rotating body. So although these metrics are adequate to describe a situation such as that of Hafele–Keating, they are not necessarily good enough to model the timing of clocks when we require accuracies of 1 ns or better over the course of several days. In fact, no solution to Einstein's equations of gravity is known that is really applicable to clocks on a rotating Earth. The "Kerr metric" might be considered: the Kerr and Schwarzschild metrics are both "vacuum solutions" of Einstein's equations, describing a universe that contains a point mass but is otherwise empty. Unlike the Schwarzschild metric, the Kerr metric allows the point mass to have angular momentum. But whereas the Schwarzschild metric also describes the gravity field external to a non-rotating spherical mass of *non-zero* radius (a result known as Birkhoff's theorem), the Kerr metric does not describe the gravity field of a rotating mass of non-zero radius.

Even so, the Kerr metric might be hoped to at least slightly describe the effect on spacetime of the rotation of a non-point mass. The metric predicts that a particle can orbit the central point mass yet carry no angular momentum. This instance of "frame dragging", also known as the Lense–Thirring Effect, has possibly been observed by the Gravity Probe B satellite, although the relevant measurements are so exceedingly difficult to make that the results of this experiment will no doubt be discussed in the relativity community for years to come.

The bottom line is that no solution to Einstein's equations is known that describes the details of timing, to a "very high" level of accuracy, of clocks fixed to a rotating Earth. The subject is subtle and not in a finished form; and while it might be thought that experiments can decide what metric is sufficient for all purposes, the philosophical difficulties in applying relativity correctly when interpreting those experiments are not universally agreed upon by the relativity community.

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19. ABSTRACT We investigate the extent to which a high-precision clock might be able to synchronise another to its own time, when they are part of a network that requires time-stamping events to a very high precision. Synchronisation at an ultra-fine level is strongly subject to the rules of relativity: clocks that are at rest in a single inertial frame can (in principle) synchronise each other to any accuracy required, whereas clocks at rest on Earth or on satellites are not inertial, and hence cannot necessarily synchronise each other to an arbitrary level of accuracy. Aside from the standard result that clocks over a wide area of Earth cannot agree on the timing of events on Earth to better than some tens of nanoseconds (a result which does not contradict the success of satellite-positioning technology), we discuss simultaneity in detail, and prove a related and new result for the extent to which two clocks at rest on Earth at the same height might agree on the meaning of "now". Although the answer requires no change in current technology, it must be understood in context. This report describes that context in detail.						