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Asymptotic Distribution of Rewards Accumulated by Alternating Renewal Processes

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ABSTRACT

This technical note considers processes that alternate randomly between 'working' and 'broken' over an interval of time. Suppose that the process is rewarded whenever it is 'working', at a rate that can vary during the time interval but is known completely. We prove that if the time interval is long then the accumulated reward is approximately normally distributed and the approximation becomes perfect as the interval becomes infinitely long. Moreover we calculate the means and variances of those normal distributions. Formally, consider an alternating renewal process on the states 'working' vs 'broken'. Suppose that during any interval $[0, \tau]$, the process is rewarded at rate $g(t/\tau)$ if it is working at time t. Let Q_τ be the reward that is accumulated during $[0, \tau]$. We calculate μ_{Q_τ} and $\sigma^2_{Q_\tau}$ such that $(Q_\tau - \mu_{Q_\tau})/\sigma_{Q_\tau}$ converges in distribution to a standard normal distribution as $\tau \to \infty$.

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November 2019: This technical note was revised to explicitly recognize that in an alternating renewal process of 'working' and 'broken' durations, the broken durations are allowed to depend on the working durations. The text on pages 1–2 was revised to tighten up assumptions and correctly quote the work by Takács. There were no other changes to the document, nor to the overall results.

Asymptotic Distribution of Rewards Accumulated by Alternating Renewal Processes

Executive Summary

This technical note documents some research into processes that alternate randomly between 'working' and 'broken' over an interval of time. It supposes that the process is rewarded whenever it is 'working', at a rate that can vary during the time interval but is known completely. We study the reward that is accumulated over that time interval. For example, consider a solar panel that can earn money if it is exposed to the sun, at a rate of 5 dollars per hour before noon and 10 dollars per hour after noon. What is the amount of money that it will earn over a given 24 hour period?

The key finding is that if the time interval is long then the accumulated reward is approximately normally distributed, and the approximation becomes perfect as the interval becomes infinitely long. The research also calculates the means and variances of those normal distributions. The values are obtained from the rates at which the process is rewarded when working (the dollars per hour in the example given above), and statistics about the times to failure and times to repair (the durations to go from working to broken and from broken to working).

This technical note is the expanded version of an article that was prepared for the journal *Statistics & Probability Letters* [Hew 2017]. It provides the details of the proofs that were abridged for the journal article. The research was motivated by studies of a number of military operations. When collapsed to their essentials, the operations could be modelled in terms of a sensor that alternates randomly between working and broken, and is looking for a target that reluctantly gives away glimpses at random times. Consider in particular the probability of seeing the k-th glimpse. Intuitively, at any time, the glimpse provides some probability of being detected *if* the sensor is working at that time. The probability of seeing the glimpse is the accumulation of those probabilities over the time interval. Hence by using the results in this article, we know that the probability of seeing the k-th glimpse is approximately normally distributed, and we can use that knowledge to make predictions about operational performance. Full details will be reported separately.

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1. Introduction

Consider a stochastic process X_t that at any given time t is either 'working' or 'broken', arising from an alternating renewal process that has working durations $\{W_k\}$ alternated with broken durations $\{B_k\}$ where $(W_1, B_1), (W_2, B_2), \ldots, (W_k, B_k), \ldots$ is a sequence of mutually independent, identically distributed, non-negative, vector random variables. We suppose that over an interval of time $[0, \tau]$, the process is rewarded at rate $g(t/\tau)$ if it is working at time t where g is a real-valued function on the interval $[0, 1]$. Let $Q_\tau = \int_0^\tau g(t/\tau) X_t \, dt$ be the *accumulated reward*, namely the reward that is accumulated by the process X_t over the time interval $[0, \tau]$ under the reward rate function g.

This technical note establishes that if the time interval $[0, \tau]$ is long then the accumulated reward Q_τ is approximately normally distributed, and the approximation becomes perfect as the interval becomes infinitely long. Moreover we calculate the means and variances of those normal distributions. Formally, let $\mathcal{P}\{\cdot\}$ denote 'probability of', $\mathbb{E}(\cdot)$ denote 'expected value of', $\mathcal{N}(\mu, \sigma^2)$ be the normal distribution with mean μ and variance σ^2, and $\Rightarrow$ denote convergence in distribution. We prove the following:

Theorem. *Let X_t be an alternating renewal process on $\{0, 1\}$ with $0 =$ 'broken', $1 =$ 'working', formed from working durations $\{W_k\}$ alternated with broken durations $\{B_k\}$ where $\{(W_k, B_k)\}$ is a sequence of mutually independent, identically distributed, non-negative, vector random variables. Recall (see text below) that there exist functions $z_1(t)$ and $z_0(t)$ such that*

$$\mathcal{P}\{X_t = 1 | X_s = 1\} = p + (1 - p) \cdot z_1(t - s)$$
$$\mathcal{P}\{X_t = 0 | X_s = 0\} = 1 - p + p \cdot z_0(t - s)$$

where $p = \frac{\beta}{\alpha+\beta}$ given $\beta = \mathbb{E}(W_k)$, $\alpha = \mathbb{E}(B_k)$. Given $g : [0, 1] \to \mathbb{R}$, put $Q_\tau = \int_0^\tau g(t/\tau) X_t \, dt$ (reward the process at rate $g(t/\tau)$ if it is working at time t), and set

$$\mu_{Q_\tau} = \bar{g} \mu_{U_\tau} \qquad \mu_{U_\tau} = p\tau$$
$$\sigma^2_{Q_\tau} = \gamma \sigma^2_{U_\tau} \qquad \sigma^2_{U_\tau} = 2p(1 - p)\tau\zeta$$

where $\bar{g} = \int_0^1 g(x) \, dx$, $\gamma = \int_0^1 (g(x))^2 \, dx$, $\zeta = \int_0^\infty z(t) \, dt$, and $z(t) = (1 - p) \cdot z_1(t) + p \cdot z_0(t)$. Suppose that all of the following conditions are satisfied:

- $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty, \mathbb{E}(B_k^3) < \infty$, *for all k.*
- $0 < \zeta < \infty$, *and there exists $\hat{z}(t)$ continuous and nonincreasing such that $|z(t)| \leq \hat{z}(t)$ for all t sufficiently large and $\int_0^\infty \hat{z}(t) \, dt < \infty$.*
- $-\infty < \bar{g} < \infty$, $0 < \gamma < \infty$, *and $|\int_0^1 g(x) g'(x) \, dx| < \infty$.*

Then $(Q_\tau - \mu_{Q_\tau})/\sigma_{Q_\tau} \Rightarrow \mathcal{N}(0, 1)$ as $\tau \to \infty$.

Remark. *If $g(x) = 1$ for all x then Q_τ equals the uptime U_τ, namely the cumulative duration in the working state during $[0, \tau]$.*

Remark. *If* $F(x) = g^{-1}(x)$ *is a well-defined cumulative distribution function, and* μ_R *and* σ_R^2 *are the mean and variance of the distribution defined by* F*, then* $\bar{g} = \mu_R$ *and* $\gamma = \sigma_R^2 + \mu_R^2$ *(see Appendix for proof).*

The finding appears to be novel in studies of alternating renewal processes, in two respects: First, the process accumulates a reward at rate g. Second, the value obtained for $\sigma_{U_\tau}^2$ is new. Indeed, we see that $\sigma_{U_\tau}^2$ is fully determined by p and ζ, where ζ comes from the process forgetting its initial conditions.

The existence of z_1 and z_0 is assured, as it is well-known [Trivedi 2002] that X_t becomes stationary from any starting condition. While it may be difficult to explicitly obtain z_1 and z_0, we can harness work by Takács [1959, 1974]: Put $\sigma_\alpha^2 = \mathbb{E}(B_k^2)$, $\sigma_\beta^2 = \mathbb{E}(W_k^2)$, and suppose that $\mathbb{E}(W_k) > 0$, $\mathbb{E}(B_k) > 0$, $\mathbb{E}(W_k^2) > 0$, $\mathbb{E}(B_k^2) > 0$. If the mutually-independent, identically distributed, vector random variables $\{(W_k, B_k)\}$ belong to the domain of normal attraction of a two-dimensional normal distribution function of type $\mathcal{N}\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1 & \rho\\ \rho & 1\end{smallmatrix}\right]\right)$ then $(U_\tau - \mu_{U_\tau})/\sigma_{U_\tau}$ converges in distribution to $\mathcal{N}(0,1)$ as $\tau \to \infty$ where

$$\mu_{U_\tau} = \frac{\beta}{\alpha+\beta}\tau$$
$$\sigma_{U_\tau}^2 = \frac{\alpha^2\sigma_\alpha^2 + \beta^2\sigma_\beta^2 - 2\rho\alpha\beta\sigma_\alpha\sigma_\beta}{(\alpha+\beta)^3}\tau$$

by [Takács 1974, Example 3]. We note in passing that $\{(W_k, B_k)\}$ belong to the domain of normal attraction of a normal distribution if and only if the joint distribution of $\{(W_k, B_k)\}$ is non-degenerate and $\mathbb{E}(B_k^2) < \infty$, $\mathbb{E}(W_k^2) < \infty$ [Encyclopedia of Mathematics 2019]; indeed ρ is the correlation of W_k with B_k (for all k as $\{(W_k, B_k)\}$ are identically distributed). Otherwise if the sequences $\{W_k\}$ and $\{B_k\}$ are independent and $\mathbb{E}(B_k^2) < \infty$, $\mathbb{E}(W_k^2) < \infty$ then the above equations apply with $\rho = 0$ [Takács 1959, Example 1].

The research in this article was motivated by studies of a number of military operations. When collapsed to their essentials, the operations could be modelled in terms of a sensor that alternates stochastically between working and broken, and is looking for a target that reluctantly gives away glimpses at random times. We consider in particular the probability of seeing the k-th glimpse. Construct g from the probability density function for the waiting time to the k-th glimpse; intuitively, during infinitesimal interval $[t, t+\delta t]$ the glimpse provides some probability of being detected *if* the sensor is working at that time. The probability of seeing the glimpse is the accumulation of those probabilities over the time interval $[0, \tau]$, namely Q_τ. Hence by using the results in this article, we know that the probability of seeing the k-th glimpse is approximately normally distributed, and we can use that knowledge to make predictions about operational performance.

The question, therefore, was about an alternating renewal process that is rewarded at some deterministic rate whenever it is working. There were no apparent results in the literature. The process studied here is *not* a renewal-reward process as usually defined. Indeed in a renewal-reward process, we have durations that are punctuated by rewards. An example is a machine that at random times, credits or debits a random amount of money from a bank account. In the process studied in this article, when the process is working it accumulates a reward at a rate that is deterministic over $[0, \tau]$. An example is a solar panel that sells electricity into a power grid, earning money at a rate that can change deterministically during the day, but where the panel is blocked during random intervals.

2. Preliminaries

This section covers the results that we will need to form our proofs. We will invoke Peligrad's central limit theorem for linear processes under strong mixing. Let $\sigma(\cdot)$ denote the σ-field generated by a collection of random variables.

Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be two σ-algebras of events and define*

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup |\mathcal{P}\{\mathrm{AB}\} - \mathcal{P}\{\mathrm{A}\}\mathcal{P}\{\mathrm{B}\}| \qquad \mathrm{A} \in \mathcal{A}, \mathrm{B} \in \mathcal{B}$$

*A sequence of random variables $Z_1, Z_2, \ldots$ is α-*mixing (strong mixing) *if $\alpha_d \to 0$ where*

$$\alpha_d = \sup_k \alpha(\sigma(Z_1, \ldots, Z_k), \sigma(Z_{k+d}, Z_{k+d+1}, \ldots))$$

Lemma 1 (Peligrad's central limit theorem for linear processes under strong mixing)**.** *Let $\{a_{nk} : 1 \le k \le n\}$ be a triangular array of real numbers such that $\sup_n \sum_{k=1}^n a_{nk}^2 < \infty$ and $\max_{1\le k\le n}|a_{nk}| \to 0$ as $n \to \infty$. Let $\{Z_k\}$ be a centred stochastic sequence such that $\{Z_k^{2+\delta}\}$ is a uniformly integrable family for some $\delta > 0$, $\inf_k \mathrm{Var}(Z_k) > 0$, and $\mathrm{Var}(\sum_{k=1}^n a_{nk}Z_k) = 1$. If $\{Z_k\}$ is α-mixing and $\sum_d d^{2/\delta}\alpha_d < \infty$ then $\sum_{k=1}^n a_{nk}Z_k \Rightarrow \mathcal{N}(0, 1)$ as $n \to \infty$.*

Proof. See Peligrad & Utev [1997, Theorem 2.2, case (c)] . While the full theorem provides conditions for ϕ-mixing, ρ-mixing, and α-mixing, we will only call on α-mixing. □

We will need the following result on the α-mixing of regenerative processes.

Lemma 2. *Let $Z_0, Z_1, Z_2, \ldots$ be a stationary regenerative process with regeneration times $0 = T_0, T_1, T_2, \ldots$. For any k, define $\tau_k = T_{k+1} - T_k$. If the process is aperiodic, positive recurrent, and $\mathbb{E}(\tau_1^K) < \infty$ then the process is strong mixing and $\alpha_d = o(d^{1-K})$.*

Proof. See Glynn's study of regenerative processes [Glynn 1982]. Theorem 6.3 shows that any aperiodic, positive recurrent, regenerative process $\{Z_k\}$ is strong mixing. There exists an 'associated process' $\{Z'_k\}$ that is stationary. Proposition 6.10 obtains the strong mixing coefficient α_d for $\{Z'_k\}$. Proposition 4.7 confirms that if $\{Z_k\}$ is stationary then $\{Z_k\}$ and $\{Z'_k\}$ have the same distribution. □

We will use the following result on the variance of linear processes.

Definition. *For any sequence of random variables $Z_1, Z_2, \ldots$, define*

$$\mathrm{b}^2_{\{Z_k\}_k} = \mathbb{E}(Z_1^2) + 2\sum_{k=1}^{\infty} \mathbb{E}(Z_1 Z_{1+k})$$

*(*b *for [Billingsley 2008, Theorem 27.4], the original source of this expression.)*

Lemma 3. *Suppose that* $Z_1, Z_2, \ldots$ *are real-valued and strictly stationary,* $\mathbb{E}(Z_k) = 0$ *for all* k*,* $g : [0,1] \to \mathbb{R}$ *and* $S_n = \sum_{k=1}^{n} g(\frac{k}{n}) Z_k$*.*

1. *If* $\sum_{k=1}^{\infty} \mathbb{E}(Z_1 Z_{1+k})$ *is absolutely convergent,* $|\int_0^1 g(x)g'(x)\,dx| < \infty$*, and* $0 < \int_0^1 (g(x))^2\,dx < \infty$*, then* $\mathrm{Var}(S_n)/(n\gamma_n) \to \mathrm{b}^2_{\{Z_k\}_k}$ *as* $n \to \infty$*, where* $\gamma_n = \frac{1}{n}\sum_{k=1}^{n}\left(g(\frac{k}{n})\right)^2$*.*

2. *If in addition* $\mathrm{b}_{\{Z_k\}_k} > 0$ *and* $S_n/\sqrt{\mathrm{Var}(S_n)} \Rightarrow \mathcal{N}(0,1)$ *as* $n \to \infty$ *then* $S_n/(\mathrm{b}_{\{Z_k\}_k}\sqrt{n\gamma_n})$ $\Rightarrow \mathcal{N}(0,1)$ *as* $n \to \infty$*.*

Proof. See Appendix. □

3. Proof of Main Result

Our core intuition is as follows: Construct

$$Q_{n,\delta t} = \left(g(\tfrac{1}{n})X_{t_1} + g(\tfrac{2}{n})X_{t_2} + \cdots + g(1)X_{t_n}\right)\cdot \delta t$$

for $t_k \in [0,\tau]$ at spacings of δt. Then $Q_{n,\delta t}$ is an approximation to Q_τ that becomes perfect as $n \to \infty$. But $Q_{n,\delta t}$ is also the sum of random variables, so we can invoke a central limit theorem. Thus the distribution of Q_τ can be approximated by a normal distribution, and the approximation becomes perfect as $\tau \to \infty$.

Ultimately, we seek to formalize our intuition as appropriate statements about convergence in distribution. On any interval $[0,\tau]$ declare

$$V_\tau = Q_\tau - \bar{g}p\tau$$

For any $\delta t > 0$ define

$$\begin{aligned} t_k &= (k-1)\cdot\delta t \\ Y_k &= X_{t_k} - p \end{aligned}$$

for $k = 1, 2, \ldots$. For any positive integer n put

$$\begin{aligned} Q_{n,\delta t} &= \left(g(\tfrac{1}{n})X_{t_1} + g(\tfrac{2}{n})X_{t_2} + \cdots + g(1)X_{t_n}\right)\cdot\delta t \quad \text{(restated)} \\ V_{n,\delta t} &= \left(g(\tfrac{1}{n})Y_1 + g(\tfrac{2}{n})Y_2 + \cdots + g(1)Y_n\right)\cdot\delta t \\ \sigma^2_{n,\delta t} &= (n\cdot\delta t)\cdot p\,(1-p)\cdot\left(\delta t + 2\sum_{k=1}^{\infty} z(t_k)\,\delta t\right) \\ \gamma_n &= \frac{1}{n}\sum_{k=1}^{n}\left(g(\tfrac{k}{n})\right)^2 \end{aligned}$$

Without loss of generality, we assume that the process is strictly stationary at time zero. For there exists s such that $z_1(s)$ and $z_0(s)$ are arbitrarily close to zero, so we may shift our analysis from $[0, \tau]$ to $[s, s+\tau]$. Shifting τ to $s+\tau$ will not matter, as we will be taking $\tau \to \infty$. Consequently Y_k is strictly stationary for all k. Moreover for all t

$$\mathcal{P}\{X_t = 1\} = p$$
$$\mathcal{P}\{X_t = 0\} = 1 - p$$

so $\mathbb{E}(Y_k) = 0$ for all k. Declare the following cumulative distribution functions

$$G_\tau(v) = \mathcal{P}\{V_\tau \leq v\}$$
$$G_{n,\delta t}(v) = \mathcal{P}\{V_{n,\delta t} \leq v\}$$
$$H(\cdot; \mu, \sigma^2) \text{ for } \mathcal{N}(\mu, \sigma^2)$$

Let $\mathbb{R}$ denote the real numbers and $\mathbb{Z}_{\geq 0}$ denote the non-negative integers.

We will prove the following propositions.

Proposition 1. *If $-\infty < \bar{g} < \infty$, then for any $v \in \mathbb{R}$, $\psi > 0$, and $\epsilon_1 > 0$ there exists $\delta t_1 > 0$ such that if $\delta t < \delta t_1$, $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$, $n = m + m'$ for any $m' \in \mathbb{Z}_{\geq 0}$, and $\tau = n \cdot \delta t$ then $|G_{n,\delta t}(v) - G_\tau(v)| < \epsilon_1$.*

Proposition 2. *If $0 < p < 1$, $0 < \zeta < \infty$, and $0 < \gamma < \infty$, then for any $v \in \mathbb{R}$, $\psi > 0$, and $\epsilon_2 > 0$ there exists $\delta t_2 > 0$ such that if $\delta t < \delta t_2$, $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$, $n = m + m'$ for any $m' \in \mathbb{Z}_{\geq 0}$ and $\tau = n \cdot \delta t$ then $0 < \sigma^2_{n,\delta t} < \infty$ and $\left| H(v; 0, \gamma_n \sigma^2_{n,\delta t}) - H(v; 0, \sigma^2_{Q_\tau}) \right| < \epsilon_2$.*

Proposition 3. *If $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty, \mathbb{E}(B_k^3) < \infty$ for all k, $0 < \zeta < \infty$, there exists $\hat{z}(t)$ continuous and nonincreasing such that $|z(t)| \leq \hat{z}(t)$ for all t sufficiently large and $\int_0^\infty \hat{z}(t)\, dt < \infty$, and $0 < \gamma < \infty$ and $|\int_0^1 g(x) g'(x)\, dx| < \infty$, then for any $v \in \mathbb{R}$, $\delta t > 0$, and $\epsilon_3 > 0$ there exists $N_3 > 0$ such that if $n > N_3$ and $0 < \sigma^2_{n,\delta t} < \infty$ then $\left| G_{n,\delta t_3}(v) - H(v; 0, \gamma_n \sigma^2_{n,\delta t_3}) \right| < \epsilon_3$.*

We will then be equipped to prove the theorem, namely:

Proposition 4. *If the conditions of the theorem are met then for all v and $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left| G_\tau(v) - H(v; 0, \sigma^2_{Q_\tau}) \right| < \epsilon$.*

In effect, we show that the diagram at Figure 1 is commutative. That is, we control the discrepancy $\epsilon = \left| G_\tau(v) - H(v; 0, \sigma^2_{Q_\tau}) \right|$ by decomposing it into

$$\epsilon_1 = |G_\tau(v) - G_{n,\delta t}(v)|$$
$$\epsilon_2 = \left| H(v; 0, \gamma_n \sigma^2_{n,\delta t}) - H(v; 0, \sigma^2_{Q_\tau}) \right|$$
$$\epsilon_3 = \left| G_{n,\delta t}(v) - H(v; 0, \gamma_n \sigma^2_{n,\delta t}) \right|$$

We choose δt to satisfy ϵ_1 and ϵ_2. We then choose n large enough to satisfy ϵ_3 *knowing that* we can do so without compromising ϵ_1 or ϵ_2.

$$\begin{array}{ccc} G_{n,\delta t}(v) & \xrightarrow[n\to\infty]{(3)} & H\left(v;0,\gamma_n\sigma^2_{n,\delta t}\right) \\ (1)\Big\downarrow{\scriptstyle \delta t\to 0} & & (2)\Big\downarrow{\scriptstyle \delta t\to 0} \\ G_\tau(v) & \xrightarrow[\tau\to\infty]{(4)} & H\left(v;0,\sigma^2_{Q_\tau}\right) \end{array}$$

Figure 1: We prove that this diagram is commutative, via Propositions 1 through 4 as labelled.

3.1. Proof of Proposition 1

Proof. Consider the functions $x_t : [0,\infty) \to \{0,1\}$ (where 0 = 'broken', 1 = 'working'). Put

$$\begin{aligned} \lambda(x_t; s) &= \int_0^s g(t/s) x_t \, dt - \bar{g}ps \\ \mu(x_t; s, \delta t) &= \sum_{k=0}^{\lfloor s/\delta t \rfloor} g(t_k/s) x_{t_k} \delta t - \bar{g}ps \qquad \text{where } t_k = k \cdot \delta t \\ a_s &= \inf_{x_t} \lambda(x_t; s) \\ b_s &= \sup_{x_t} \lambda(x_t; s) \end{aligned}$$

Put

$$\begin{aligned} \chi(v; s, \delta t) &= \{x_t : \lambda(x_t; s) = v \text{ but } \mu(x_t; s, \delta t) \neq v \text{ or } \lambda(x_t; s) \neq v \text{ but } \mu(x_t; s, \delta t) = v\} \\ \xi(v; \delta t) &= \sup_{\kappa \in [1,\infty)} \mathcal{P}\{X_t \in \chi(\kappa v; \kappa\psi, \delta t)\} \end{aligned}$$

(*Intuitively*: Define X_t as being *stretched* by factor κ if the sojourns in each state are each multiplied by κ. If X_t yields reward v over $[0,\psi]$ then stretching it by κ will yield κv over $[0,\kappa\psi]$. Hence $\xi(v;\delta t)$ is the supremum probability of getting a realization of X_t where the actual reward is v but the approximated value is different, where the supremum is taken over all possible stretchings of the time window $[0,\psi]$.)

For all $v \in [a_\psi, b_\psi]$, $0 \leq \xi(v;\delta t) \leq 1$ for all δt (from being a supremum over probabilities), and $\xi(v;\delta t) \to 0$ as $\delta t \to 0$ (pointwise convergence). By the bounded convergence theorem, there exists δt_1 such that if $\delta t < \delta t_1$ then

$$\int_{[a_\psi, b_\psi]} \xi(v;\delta t)\, dv < \epsilon_1$$

Hence if $\delta t < \delta t_1$ then

$$\begin{aligned} |G_{m+m',\delta t}(v') - G_\tau(v')| &= \int_{a_\psi}^{\frac{\psi}{\tau}v'} \mathcal{P}\{X_t \in \chi(\tfrac{\tau}{\psi}v; \tau, \delta t)\}\, dv \\ &\leq \int_{a_\psi}^{b_\psi} \xi(v;\delta t)\, dv \\ &< \epsilon_1 \end{aligned}$$

$\square$

3.2. Proof of Proposition 2

We prove in three steps:

1. *If $0 < p < 1$ and $0 < \zeta < \infty$ then for any $v \in \mathbb{R}$, $\psi > 0$, and $\epsilon_2 > 0$ there exists $\delta t_2 > 0$ such that if $\delta t < \delta t_2$, $n \in \mathbb{Z}_{\geq 0}$ and $\tau = n \cdot \delta t$ then $0 < \sigma^2_{n,\delta t} < \infty$.*

   Proof. We have

   $$\zeta = \int_0^\infty z(t)\, dt = \lim_{\delta t \to 0} \sum_{k=1}^{\infty} z(t_k)\, \delta t$$

   by definition, so there exists $\delta t_2 > 0$ such that the series $\sum_{k=1}^{\infty} z(t_k)\, \delta t$ is convergent for all $\delta t < \delta t_2$. Moreover $\zeta > 0$ so we can refine δt_2 so that $\sum_{k=1}^{\infty} z(t_k)\, \delta t > 0$ whenever $\delta t < \delta t_2$. Likewise $\gamma > 0$ so we can again refine δt_2 so that $\gamma_n > 0$ whenever $\delta t < \delta t_2$. Hence $0 < \sigma^2_{n,\delta t} < \infty$ whenever $\delta t < \delta t_2$. □

2. *For any $v \in \mathbb{R}$, $\psi > 0$, and $\epsilon_2 > 0$ there exists $\delta t_2 > 0$ such that if $\delta t < \delta t_2$ and $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$ then $\left| H\big(v; 0, \gamma_m \sigma^2_{m,\delta t}\big) - H\big(v; 0, \gamma_m \sigma^2_{U_\psi}\big) \right| < \epsilon_2$.*

   Proof. $H\big(\cdot;\cdot,\sigma^2\big)$ is continuous for all $\sigma^2 > 0$, so it is sufficient to show that $\sigma^2_{m,\delta t} \to \sigma^2_{U_\psi}$ as $\delta t \to 0$. But this is evident given the equation for ζ at step 1, and $m \cdot \delta t \to \psi$ as $\delta t \to 0$. □

3. *If $\tau = (m + m') \cdot \delta t$ then $\left| H\big(v; 0, \gamma_{m+m'} \sigma^2_{m+m',\delta t}\big) - H\big(v; 0, \sigma^2_{Q_\tau}\big) \right| < \epsilon_2$.*

   Proof. As for Proposition 1. □

3.3. Proof of Proposition 3

We start with the following lemma.

Lemma 4. *(i) $\sum_{k=1}^{\infty} \mathbb{E}(Y_1 Y_{1+k})$ is absolutely convergent if and only if $\sum_{k=1}^{\infty} |z(t_k)|$ is (absolutely) convergent. (ii) $\mathrm{b}^2_{\{Y_k\}_k} n = \sigma^2_{n,\delta t}$ for all n.*

Proof. We have

$$\begin{aligned} \mathbb{E}(Y_1^2) &= (\delta t)^2 \cdot \mathbb{E}((X_{t_1} - p)^2) \\ &= (\delta t)^2 \cdot \Big((1-p)^2 \cdot p + (-p)^2 \cdot (1-p) \Big) \\ &= (\delta t)^2\, p\,(1-p) \end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Y_1 Y_{1+k}) &= (\delta t)^2\,(1-p)\,(1-p)\,p\cdot(p+(1-p)\cdot z_1(t_k)) + \\
&\quad (\delta t)^2\,(1-p)\,(-p)\,p\cdot(1-p-(1-p)\cdot z_1(t_k)) + \\
&\quad (\delta t)^2\,(-p)\,(1-p)\,(1-p)\cdot(p-p\cdot z_0(t_k)) + \\
&\quad (\delta t)^2\,(-p)\,(-p)\,(1-p)\cdot(1-p+p\cdot z_0(t_k)) \\
&= (\delta t)^2\,p\,(1-p)\,\big((1-p)\,p - p\,(1-p) \\
&\qquad\qquad -p\,(1-p) + p\,(1-p)\big) + \\
&\quad (\delta t)^2\,p\,(1-p)\,\big((1-p+p)\,(1-p)\cdot z_1(t_k) + \\
&\qquad\qquad (1-p+p)\,p\cdot z_0(t_k)\big) \\
&= (\delta t)^2\,p\,(1-p)\,((1-p)\cdot z_1(t_k) + p\cdot z_0(t_k)) \\
&= (\delta t)^2\,p\,(1-p)\cdot z(t_k)
\end{aligned}$$

so $\sum_{k=1}^{\infty}|\mathbb{E}(Y_1 Y_{1+k})| = (\delta t)^2\,p\,(1-p)\sum_{k=1}^{\infty}|z(t_k)|$ and thus each series is absolutely convergent if and only if the other one is. Moreover

$$\begin{aligned}
\mathrm{b}^2_{\{Y_k\}_k} &= \delta t\cdot p\,(1-p)\cdot\left(\delta t + 2\sum_{k=1}^{\infty} z(t_k)\delta t\right) \\
\mathrm{b}^2_{\{Y_k\}_k} n &= \sigma^2_{n,\delta t}
\end{aligned}$$

□

We are now equipped to prove the proposition. For any k and n, put

$$\begin{aligned}
a'_{nk} &= g(\tfrac{k}{n})\cdot\delta t \\
a_{nk} &= \frac{a'_{nk}}{\sqrt{\mathrm{Var}(\sum_{k=1}^{n} a'_{nk} Y_k)}}
\end{aligned}$$

Then $V_{n,\delta t} = \sum_{k=1}^{n} a'_{nk} Y_k$ by construction. In steps 1 through 11, we verify that $\{Y_k\}$ satisfies the conditions of Peligrad's central limit theorem. Then step 12 obtains Proposition 3.

1. *$Y_1, Y_2, \ldots$ is strictly stationary, $\mathbb{E}(Y_k) = 0$.*

   Proof. By construction. □

2. *$\sum_{k=1}^{\infty}|z(t_k)|$ is absolutely convergent.*

   Proof. There exists a continuous, nonincreasing function $\hat{z}(t)$ such that $|z(t)| \le \hat{z}(t)$ for all t sufficiently large and $\int_0^{\infty}\hat{z}(t)\,dt < \infty$. So $\sum_{k=1}^{\infty}|\hat{z}(t_k)| < \infty$ and hence $\sum_{k=1}^{\infty}|z(t_k)| < \infty$. □

3. *The conditions for Lemma 3 are satisfied with $0 < \mathrm{b}^2_{\{Z_k\}_k} < \infty$.*

Proof. $\{Y_k\}$ is strictly stationary and centred (step 1), and $\sum_{k=1}^{\infty} \mathbb{E}(Y_1 Y_{1+k})$ is absolutely convergent (step 2 and Lemma 4). Now $|\int_0^1 g(x)g'(x)\,dx| < \infty$, $0 < \int_0^1 (g(x))^2\,dx < \infty$, and $0 < \sigma^2_{n,\delta t} < \infty$ by assumption. Hence $0 < \mathrm{b}^2_{\{Z_k\}_k} < \infty$ by Lemma 4. □

4. $\sup_n \sum_{k=1}^n a_{nk}^2 < \infty$.

   Proof. By Lemma 3 (via step 3), we have $\sum_{k=1}^n a'_{nk} Y_k = S_n \cdot \delta t$ so

$$\sum_{k=1}^n a_{nk}^2 = \frac{\sum_{k=1}^n (g(\frac{k}{n}) \cdot \delta t)^2}{\mathrm{Var}(S_n \cdot \delta t)} = \frac{n\gamma_n}{\mathrm{Var}(S_n)} \to \frac{1}{\mathrm{b}^2_{\{Z_k\}_k}}$$

   as $n \to \infty$, and $0 < 1/\mathrm{b}^2_{\{Z_k\}_k} < \infty$ (again by step 3). □

5. $\max_{1 \le k \le n} |a_{nk}| \to 0$ *as* $n \to \infty$.

   Proof. By the assumptions on g, there exists $x \in [0,1]$ such that $g(\frac{k}{n})^2 \le g(x)^2$ for all $1 \le k \le n$. Now in terms of Lemma 3, for any n and $1 \le k \le n$

$$a_{nk}^2 \le \frac{(g(x) \cdot \delta t)^2}{\mathrm{Var}(S_n \cdot \delta t)} = \frac{1}{n} \cdot \frac{g(x)^2}{\gamma_n} \cdot \frac{n\gamma_n}{\mathrm{Var}(S_n)}$$

   Now by Lemma 3 (via step 3), the right hand side $\to 0$ as $n \to \infty$. □

6. $\mathbb{E}(Y_k^{2+\delta}) < \infty$ *and* > 0 *for any* $\delta \ge 0$.

   Proof. We have

$$\begin{aligned}\mathbb{E}(|Y_k|^{2+\delta}) &= (\delta t)^{2+\delta} \cdot \mathbb{E}(|X_{t_k} - p|^{2+\delta}) \\ &= (\delta t)^{2+\delta} \cdot \Big(|1-p|^{2+\delta} \cdot p + |-p|^{2+\delta} \cdot (1-p)\Big) \\ &< \infty \text{ and } > 0\end{aligned}$$

   □

7. *If* $\delta = 4$ *then* $\{|Y_k|^{2+\delta}\}$ *is a uniformly integrable family.*

   Proof. By step 6, choosing $\delta = 4$ for definiteness. □

8. $\inf_k \mathrm{Var}(Y_k) > 0$.

   Proof. By step 6. □

9. $\mathrm{Var}(\sum_{k=1}^n a_{nk} Y_k) = 1$.

Proof. Immediate by construction of a_{nk}. □

10. *$\{Y_k\}$ is strong mixing with $\alpha_d = o(d^{-2})$.*

Proof. We apply Lemma 2: $\{Y_k\}$ is strictly stationary (step 1) and regenerative (from being a renewal process). Now $\mathbb{E}(W_k^2)+\mathbb{E}(B_k^2) > 0$ so $\{Y_k\}$ is aperiodic, and $\mathbb{E}(W_k), \mathbb{E}(B_k) < \infty$ so $\{Y_k\}$ is positive recurrent. Finally $\mathbb{E}(W_k^3), \mathbb{E}(B_k^3) < \infty$ so $\alpha_d = o(d^{-2})$. □

11. *$\sum_d d^{2/\delta}\alpha_d < \infty$ when $\delta = 4$.*

Proof. Immediate from $\alpha_d = o(d^{-2})$ (step 10). □

12. *For any $v \in \mathbb{R}$, $\delta t > 0$, and $\epsilon_3 > 0$ there exists $N_3 > 0$ such that if $n > N_3$ and $0 < \sigma^2_{n,\delta t} < \infty$ then $\left|G_{n,\delta t}(v) - H\left(v; 0, \gamma_n \sigma^2_{n,\delta t}\right)\right| < \epsilon_3$.*

Proof. By steps 1 through 11 and Lemma 1,

$$\frac{V_{n,\delta t}}{\sqrt{\mathrm{Var}(V_{n,\delta t})}} = \sum_{k=1}^{n} a_{nk} Y_k \Rightarrow \mathcal{N}(0,1)$$

as $n \to \infty$. Then by Lemma 3 (via step 3) $V_{n,\delta t}/(\sigma_{n,\delta t}\sqrt{\gamma_n}) \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$. That is, for any $v \in \mathbb{R}$ and $\epsilon_3 > 0$ there exists $N_3 > 0$ such that if $n > N_3$ then

$$\left|\mathcal{P}\left\{\frac{V_{n,\delta t}}{\sigma_{n,\delta t}\sqrt{\gamma_n}} \leq \frac{v}{\sigma_{n,\delta t}\sqrt{\gamma_n}}\right\} - H\left(\frac{v}{\sigma_{n,\delta t}\sqrt{\gamma_n}}; 0, 1\right)\right| < \epsilon_3$$

But this is true if and only if

$$\left|G_{n,\delta t}(v) - H\left(v; 0, \gamma_n \sigma^2_{n,\delta t}\right)\right| < \epsilon_3$$

□

3.4. Proof of Proposition 4

Proof. H is continuous on $\mathbb{R}$, so we are proving the proposition for any $v \in \mathbb{R}$. Choose $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon$. Choose $\psi > 0$ arbitrary and observe:

By Proposition 1: There exists $\delta t_1 > 0$ such that if $\delta t < \delta t_1$, $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$, $n = m + m'$ for any $m' \in \mathbb{Z}_{\geq 0}$, and $\tau = n \cdot \delta t$, then

$$|G_{n,\delta t}(v) - G_\tau(v)| < \epsilon_1 \tag{1}$$

Now $0 < p < 1$ from $W_k, B_k > 0$ for all k. We have $0 < \zeta < \infty$ from the assumptions about $z(t)$. Hence:

By Proposition 2: There exists $\delta t_2 > 0$ such that if $\delta t < \delta t_2$, $n = m + m'$ for any $m' \in \mathbb{Z}_{\geq 0}$, and $\tau = n \cdot \delta t$, then $0 < \sigma^2_{n,\delta t} < \infty$ and

$$\left|H\left(v; 0, \gamma_n \sigma^2_{n,\delta t}\right) - H\left(v; 0, \sigma^2_{Q_\tau}\right)\right| < \epsilon_2 \tag{2}$$

Put $\delta t' = \min\{\delta t_1, \delta t_2\}$. Then $0 < \sigma^2_{n,\delta t'} < \infty$ for all $n \geq m$ (by Proposition 2). Furthermore $z(t)$ satisfies the conditions for Proposition 3 so

By Proposition 3: There exists $N_3 > 0$ such that if $n > N_3$ then

$$\left|G_{n,\delta t'}(v) - H\left(v; 0, \gamma_n \sigma^2_{n,\delta t'}\right)\right| < \epsilon_3 \tag{3}$$

Set $\tau' = \max\{N_3, m\} \cdot \delta t'$. Now suppose that $\tau > \tau'$. Set $n = \left\lceil \frac{\tau}{\delta t'} \right\rceil$ and note that $n > N_3$ and $n = m + m'$ for some $m' \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}\left|G_\tau(v) - H\left(v; 0, \sigma^2_{Q_\tau}\right)\right| &\leq \left|G_\tau(v) - G_{n,\delta t'}(v)\right| + \\ &\quad \left|G_{n,\delta t'}(v) - H\left(v; 0, \gamma_n \sigma^2_{n,\delta t'}\right)\right| + \\ &\quad \left|H\left(v; 0, \sigma^2_{Q_\tau}\right) - H\left(v; 0, \gamma_n \sigma^2_{n,\delta t'}\right)\right| \\ &< \epsilon_1 + \epsilon_3 + \epsilon_2\end{aligned}$$

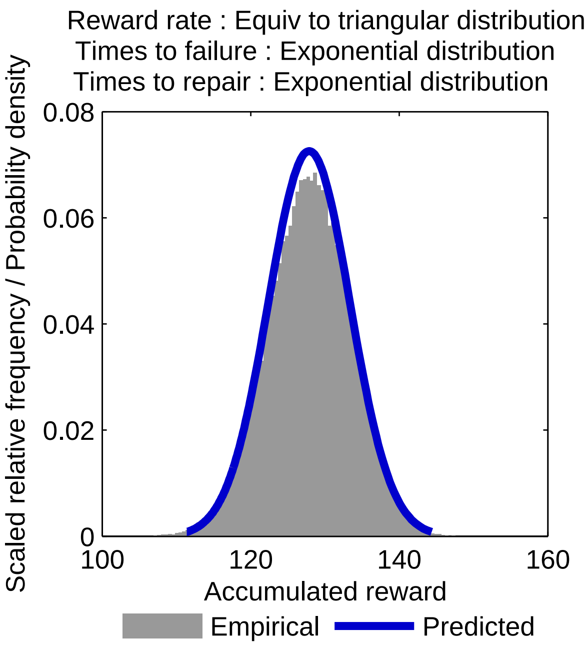
as required. □

4. Remarks

Figures 2 shows examples from experiments. In each case, the predicted distribution appears to be a good approximation to the empirical distribution.

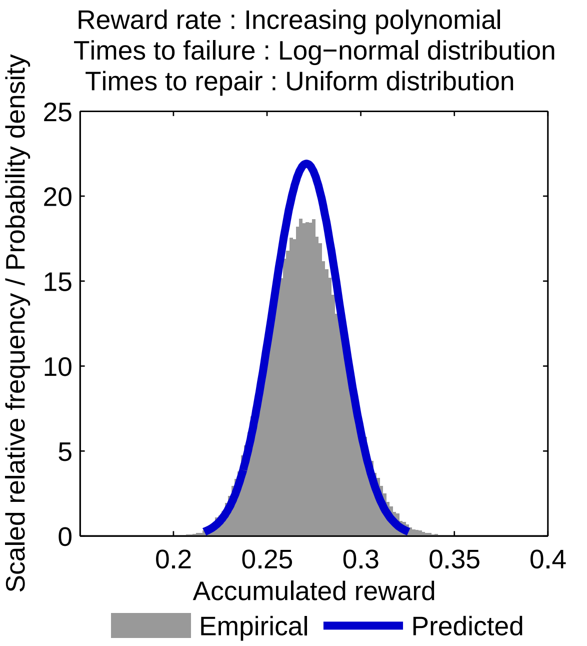
The author conjectures that the condition $\mathbb{E}(B_k^3)$, $\mathbb{E}(W_k^3) < \infty$ could be weakened to $\mathbb{E}(B_k^2)$, $\mathbb{E}(W_k^2) < \infty$. This would match the assumption made by Takács [1959, 1974]. The condition $\mathbb{E}(B_k^3)$, $\mathbb{E}(W_k^3) < \infty$ is used only to enforce α-mixing at the rate required by Peligrad's central limit theorem. Her theorem also holds under ϕ-mixing, and Glynn [1982, Theorem 6.3] states a sufficient condition for a regenerative process to be ϕ-mixing, but the present author was unable to prove that $\mathbb{E}(B_k^2)$, $\mathbb{E}(W_k^2) < \infty$ would satisfy Glynn's condition.

a)



$g(t) = F^{-1}\left(\frac{t}{\tau}\right)$ where F is the cumulative distribution function for the triangular distribution on [-1, 3] with mode at 2.
Times to failure : Exponential distribution with mean time to failure 0.7.
Times to repair : Exponential distribution with mean time to repair 0.3.

b)



$g(t) = \frac{k}{\tau}\left(\frac{t}{\tau}\right)^{k-1}$ where $k = 3$.
Times to failure : Log-normal distribution with $\mu = 0.6, \sigma = 0.2$.
Times to repair : Uniform distribution on $[3, 7]$.

Figure 2: Predicted probability density function for Q_τ vs empirical scaled-relative frequency histogram from 100,000 simulations, where $\tau = 137$ and $\delta t = \tau/1000$.

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Appendix A. The Variance of Asymptotically Normal Sums of Strictly Stationary Processes under Weighting

Let S_n be the sum of n random variables. Many central limit theorems establish that under specified conditions, $S_n/\sqrt{\mathrm{Var}(S_n)} \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$ (converges in distribution to the normal distribution with mean 0, variance 1). It can be desirable to calculate σ^2 and $f(n)$ such that $S_n/(\sigma\sqrt{n \cdot f(n)}) \Rightarrow \mathcal{N}(0,1)$. In this appendix, we prove the following:

Lemma 5. *Suppose that $Z_1, Z_2, \ldots$ are real-valued and strictly stationary, $\mathbb{E}(Z_k) = 0$ for all k, $g : [0,1] \to \mathbb{R}$ and $S_n = \sum_{k=1}^{n} g(\frac{k}{n})Z_k$. If $\sum_{k=1}^{\infty} \mathbb{E}(Z_1 Z_{1+k})$ is absolutely convergent, $|\int_0^1 g(x)g'(x)\,dx| < \infty$, and $0 < \int_0^1 (g(x))^2\,dx < \infty$, then*

$$\lim_{n\to\infty} \frac{\mathrm{Var}(S_n)}{n\gamma_n} = \sigma^2 \triangleq \mathbb{E}(Z_1^2) + 2\sum_{k=1}^{\infty} \mathbb{E}(Z_1 Z_{1+k})$$

where $\gamma_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left(g(\tfrac{k}{n})\right)^2$.

Corollary 1. *If Lemma 5 is satisfied with $\sigma > 0$ and $S_n/\sqrt{\mathrm{Var}(S_n)} \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$ then $S_n/(\sigma\sqrt{n\gamma_n}) \Rightarrow \mathcal{N}(0,1)$ as $n \to \infty$.*

Remark 1. *If $F(x) = g^{-1}(x)$ is a well-defined cumulative distribution function, and μ_R and σ_R^2 are the mean and variance of the distribution defined by F, then $\mu_R = \int_0^1 g(x)\,dx$ and $\sigma_R^2 + \mu_R^2 = \int_0^1 (g(x))^2\,dx$.*

A.1. Proofs

Proof of Lemma 5. (The following proof is derived from [Billingsley 2008, Theorem 27.4], with extensions to handle g.) Put $\rho_k = \mathbb{E}(Z_1 Z_{1+k})$, $g_k = g(\frac{k}{n})$. Now $\mathbb{E}(Z_k) = 0$ so $\mathbb{E}(S_n) = 0$ hence

$$\begin{aligned}
\mathrm{Var}(S_n) &= \mathbb{E}((g_1 Z_1 + \cdots + g_n Z_n)^2) \\
&= g_1^2\mathbb{E}(Z_1^2) + 2g_1g_2\mathbb{E}(Z_1Z_2) + \cdots + 2g_1g_{n-1}\mathbb{E}(Z_1Z_{n-1}) + 2g_1g_n\mathbb{E}(Z_1Z_n) + \\
&= g_2^2\mathbb{E}(Z_2^2) + 2g_2g_3\mathbb{E}(Z_2Z_3) + \cdots + 2g_2g_n\mathbb{E}(Z_2Z_n) + \\
&\quad \vdots \\
&\quad g_{n-1}^2\mathbb{E}(Z_{n-1}^2) + 2g_{n-1}g_n\mathbb{E}(Z_{n-1}Z_n) + \\
&\quad g_n^2\mathbb{E}(Z_n^2) \\
&= n\gamma_n\rho_0 + 2\sum_{k=1}^{n-1} \rho_k \sum_{i=1}^{n-k} g_i g_{i+k}
\end{aligned}$$

as $Z_1, Z_2, \ldots$ is strictly stationary. Then

$$\begin{aligned}
\frac{\mathrm{Var}(S_n)}{n\gamma_n} &= \rho_0 + 2\sum_{k=1}^{n-1}\rho_k \frac{1}{n\gamma_n}\sum_{i=1}^{n-k} g_i g_{i+k} \\
\left|\frac{\mathrm{Var}(S_n)}{n\gamma_n} - \sigma^2\right| &= 2\left|\sum_{k=n}^{\infty}\rho_k + \sum_{k=1}^{n-1}\left(1 - \frac{1}{n\gamma_n}\sum_{i=1}^{n-k} g_i g_{i+k}\right)\rho_k\right| \\
&= 2\left|\sum_{k=n}^{\infty}\rho_k + \sum_{k=1}^{n-1}\frac{\sum_{i=1}^{n} g_i^2 - \sum_{i=1}^{n-k} g_i g_{i+k}}{n\gamma_n}\rho_k\right| \\
&= 2\left|\sum_{k=n}^{\infty}\rho_k + \sum_{k=1}^{n-1}\frac{\sum_{i=n-k+1}^{n} g_i^2 - \sum_{i=1}^{n-k}(g_i g_{i+k} - g_i^2)}{n\gamma_n}\rho_k\right| \\
&= 2\left|\sum_{k=n}^{\infty}\rho_k + \sum_{k=1}^{n-1}\frac{\alpha_k + \beta_k}{\gamma_n}\frac{k}{n}\rho_k\right|
\end{aligned}$$

where $\alpha_k = -\frac{1}{n}\sum_{i=1}^{n-k} g_i \frac{g_{i+k} - g_i}{k/n}$, $\beta_k = \frac{1}{k}\sum_{i=n-k+1}^{n} g_i^2$. Construct $\alpha(s) = \int_0^s g(x)g'(x)\,dx$ and $\beta(s) = \int_s^1 (g(x))^2\,dx$, then $\alpha(\frac{k}{n}) \approx \alpha_k$ and $\beta(\frac{k}{n}) \approx \beta_k$ for any $k < n$. So if $\alpha^* = \sup_{s\in[0,1]}|\alpha(s)|$ and $\beta^* = \sup_{s\in[0,1]}|\beta(s)|$ then

$$\left|\frac{\mathrm{Var}(S_n)}{n\gamma_n} - \sigma^2\right| \leq 2\sum_{k=n}^{\infty}|\rho_k| + \frac{\alpha^* + \beta^* + \epsilon}{n\gamma_n}\sum_{k=1}^{n-1} k|\rho_k|$$

for some small error term ϵ where $\epsilon \to 0$ as $n \to \infty$. Moreover

$$\begin{aligned}
\sum_{k=1}^{n-1} k|\rho_k| &= \begin{array}{ccccccccc}
|\rho_1| & + & |\rho_2| & + & |\rho_3| & +\cdots+ & |\rho_{n-1}| & + \\
 & & |\rho_2| & + & |\rho_3| & +\cdots+ & |\rho_{n-1}| & + \\
 & & & & |\rho_3| & +\cdots+ & |\rho_{n-1}| & + \\
 & & & & & \vdots & & \\
 & & & & & + & |\rho_{n-1}| &
\end{array} \\
&= \sum_{i=1}^{n-1}\sum_{k=i}^{n-1}|\rho_k| \\
&\leq \sum_{i=1}^{n-1}\sum_{k=i}^{\infty}|\rho_k|
\end{aligned}$$

so

$$\left|\frac{\mathrm{Var}(S_n)}{n\gamma_n} - \sigma^2\right| \leq 2\sum_{k=n}^{\infty}|\rho_k| + \frac{\alpha^* + \beta^* + \epsilon}{n\gamma_n}\sum_{i=1}^{n-1}\sum_{k=i}^{\infty}|\rho_k|$$

To complete the proof, we show that right-hand side converges to zero as $n \to \infty$. In three steps:

1. $\sum_{k=1}^{\infty}\rho_k$ is absolutely convergent, so $\sum_{k=n}^{\infty}|\rho_k| \to 0$ as $n \to \infty$.

2. We have $\alpha^*, \beta^* < \infty$, $0 < \lim_{n\to\infty}\gamma_n < \infty$ by the assumptions about g. Specifically: if a function is integrable on $[0,1]$ then for any s it is integrable on the subintervals $[0,s]$ and $[s,1]$. Thus $\alpha(s)$ and $\beta(s)$ are continuous on $[0,1]$, hence they are bounded on $[0,1]$.

3. Put $\zeta_i = \sum_{k=i}^{\infty}|\rho_k|$ and $\omega_{n-1} = \frac{1}{n-1}\sum_{i=1}^{n-1}\zeta_i$. Now $\{\zeta_i\}_i$ is decreasing so $\omega_n \to 0$ as $n \to \infty$. Hence
$$\frac{1}{n}\sum_{i=1}^{n-1}\sum_{k=i}^{\infty}|\rho_k| = \frac{1}{n}\sum_{i=1}^{n-1}\zeta_i = \frac{n-1}{n}\omega_{n-1} \ \to 0$$
as $n \to \infty$.

□

Proof of Corollary 1. We have
$$\frac{S_n}{\sigma\sqrt{n\gamma_n}} = \frac{S_n}{\sqrt{\text{Var}(S_n)}} \cdot \frac{\sqrt{\text{Var}(S_n)}}{\sigma\sqrt{n\gamma_n}}$$
So if $S_n/\sqrt{\text{Var}(S_n)} \Rightarrow \mathcal{N}(0,1)$ and Lemma 5 is satisfied with $\sigma > 0$, then the right hand side converges in distribution to $\mathcal{N}(0,1)$ by Slutsky's theorem. □

Proof of Remark 1.

1. $\mu_R = \int_{g(0)}^{g(1)} x\,dF(x)$ by definition. Now $\int x\,dF(x) = xF(x) - \int F(x)\,dx$ and
$$\begin{aligned}\int F(x)\,dx &= \int g^{-1}(x)\,dx \\ &= \int tg'(t)\,dt && \text{via } x = g(t) \\ &= \left[tg^{-1}(t) - \int g^{-1}(t)\,dt\right] \\ &= xF(x) - \int g(x)\,dx\end{aligned}$$
which yields
$$xF(x) - \int F(x)\,dx = \int g(x)\,dx$$
Hence $\mu_R = \int_0^1 g(x)\,dx$.

2. $\sigma_R^2 + \mu_R^2 = \int_{g(0)}^{g(1)} x^2\,dF(x)$ by definition. Now $\int x^2\,dF(x) = x^2F(x) - 2\int xF(x)\,dx$ and
$$\begin{aligned}\int xF(x)dx &= \int xg^{-1}(x)\,dx \\ &= \int g(t)tg'(t)\,dt && \text{via } x = g(t) \\ &= \left[g(t)tg(t) - \int g(t)\left(g(t) + tg'(t)\right)\,dt\right] \\ &= \left[(g(t))^2\,t - \int (g(t))^2\,dt - \int g(t)tg'(t)\,dt\right]\end{aligned}$$

so

$$
\begin{aligned}
2\int g(t)tg'(t)\,dt &= \left[(g(t))^2\,t - \int (g(t))^2\;dt\right] \\
2\int xF(x)dx &= x^2g^{-1}(x) - \int (g(x))^2\;dx \\
&= x^2F(x) - \int (g(x))^2\;dx
\end{aligned}
$$

which yields

$$
x^2F(x) - 2\int xF(x)\,dx = \int (g(x))^2\;dx
$$

Hence $\sigma_R^2 + \mu_R^2 = \int_0^1 (g(x))^2\;dx$.

□

A.2. Remarks

If $g(x) = 1$ for all x then Lemma 5 reduces to the result obtained by Billingsley [2008, Theorem 27.4] and Durrett [2004, Theorem 7.8]. Billingsley and Durrett made additional assumptions that lead to σ^2 being well-defined and correct *and* asymptotic normality of S_n. The present author has extracted the assumptions and logic for σ^2 so that it stands on its own, in a form that can be used with other central limit theorems, and extended Billingsley's proof to handle g.

If in addition to being identically distributed, the variables $Z_1, Z_2, \ldots$ are independent, then $\sigma^2 = \mathbb{E}(Z_1^2)$ as per the classical Lindeberg–Lévy central limit theorem. If they are m-dependent then $\sigma^2 = \mathbb{E}(Z_1^2) + 2\sum_{k=1}^{m}\mathbb{E}(Z_1Z_{1+k})$, matching the calculations in the central limit theorem for m-dependent sequences by Hoeffding & Robbins [Theorem 2, 1948], [1985]. The author conjectures that the calculations of variance made by Hoeffding & Robbins and Ibraginov [1975, Theorem 2.2] could be extracted in the same way as was done here.

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probability theory: asymptotically normal, stochastic processes: alternating renewal process, reward

19. ABSTRACT

This technical note considers processes that alternate randomly between ‘working’ and ‘broken’ over an interval of time. Suppose that the process is rewarded whenever it is ‘working’, at a rate that can vary during the time interval but is known completely. We prove that if the time interval is long then the accumulated reward is approximately normally distributed and the approximation becomes perfect as the interval becomes infinitely long. Moreover we calculate the means and variances of those normal distributions. Formally, consider an alternating renewal process on the states ‘working’ vs ‘broken’. Suppose that during any interval $[0, \tau]$, the process is rewarded at rate $g(t/\tau)$ if it is working at time t. Let Q_τ be the reward that is accumulated during $[0, \tau]$. We calculate μ_{Q_τ} and $\sigma^2_{Q_\tau}$ such that $(Q_\tau - \mu_{Q_\tau})/\sigma_{Q_\tau}$ converges in distribution to a standard normal distribution as $\tau \to \infty$.
Revised 2019: Summary of changes at Executive Summary.