UNCLASSIFIED

Australian Government
Department of Defence
Science and Technology

An Intermittent Sensor versus a Target that Emits Glimpses as a Homogenous Poisson Process

Patrick Chisan Hew

Joint and Operations Analysis Division
Defence Science and Technology Group

DST-Group–TN–1765

Revised 2019: Summary of changes are on the imprint page.

ABSTRACT

This technical note considers a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time, the sensor has a probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in that interval. We devise closed-form approximations to the distributions for those probabilities, prove that the approximations become perfect as the time interval becomes infinitely long (asymptotic distributions, pointwise convergence), and compare the approximations with empirical results obtained from simulations.

RELEASE LIMITATION

Approved for Public Release

UNCLASSIFIED

Produced by

Joint and Operations Analysis Division
506 Lorimer St,
Fishermans Bend, Victoria 3207, Australia

Telephone: 1300 333 362

April 2018 Original release
November 2019 Reissued with corrections

November 2019 **:** This report was revised to explicitly recognize that in an alternating renewal process of 'working' and 'broken' durations, the broken durations are allowed to depend on the working durations. The text in Section 2.1 and the statement and proof of Lemma 1 were revised to tighten up the assumptions and correctly quote the work by Takács. Extensive but superficial changes were also made to remove figures that were unhelpful and cite additional literature, to match with the journal article that was published after the first version of the report. The report's overall findings were not affected.

An Intermittent Sensor versus a Target that Emits Glimpses as a Homogenous Poisson Process

Executive Summary

The work in this report was motivated by studies of operations to understand the performance that may be required from future systems; for example, unmanned aerial vehicles hunting for time-sensitive targets and submarines standing off from counter-detection. When collapsed to their essentials, the operations could be modelled in terms of a sensor that alternates between working and broken at random times, and is looking for a target that reluctantly gives away glimpses as a homogenous Poisson process. We refer to this situation as the *intermittent sensor homogenous glimpses model.*

In studies of such operations, the key measures of performance include the probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in the time interval. This report establishes that if we accept the intermittent sensor homogenous glimpses model then the measures of performance have approximations that are easy to calculate, and the approximations are close to reality when the time interval is long. So while the model is evidently an abstraction of real life, it can be sufficiently valid for a first, 'back of the envelope' analysis. Moreover the approximations provide insight into how performance will behave overall, something that can be difficult to obtain from stochastic simulation only.

The results can be applied to analysis of operations whenever the intermittent sensor homogenous glimpses model is a valid abstraction of the operation being studied. The analysis proves that the approximations become perfect in the technical sense of 'pointwise convergence' and uses simulation to compare the approximations with reality. The report will be of interest to analysts who are considering the intermittent sensor homogenous glimpses model for their work.

This page is intentionally blank

Contents

Figures

Notation

	Detection of glimpses
$[0,\tau]$	Time interval
U	Cumulative duration that the sensor is working during $[0,\tau]$
μ_U	Mean of U
σ_U^2	Variance of U
z	Mean time between glimpses from the target
ζ	Mean rate at which the target gives away glimpses $= \frac{1}{z}$
$\hat{\zeta}$	Upper bound on ζ
	Sensor reliability
W_k	Time to failure on the k-th cycle
μ_{F}	Mean of sensor's times to failure
σ_{F}^2	Variance in times to failure
B_k	Time to repair the failure in the k-th cycle
μ_{R}	Mean of times to repair the sensor
σ_{R}^2	Variance in times to repair
ρ	Correlation of W_k with B_k (for all k)
c	Calculation parameter (Lemma 1)
w	Calculation parameter (Lemma 1)
	Measures of performance during an interval $[0,\tau]$
P_n	Probability of detecting n glimpses
P^k	Probability of detecting the k-th glimpse
P_n^k	Probability of detecting the k-th glimpse given n glimpses
	Conventional formalisms
$\mathbb{R}$	Real numbers
$\mathbb{P}(\cdot)$	Probability of $\cdot$
$\mathbb{E}(\cdot)$	Expected value of $\cdot$
$\mathcal{N}(\mu,\sigma^2)$	Normal distribution with mean μ and variance σ^2
$\ln\mathcal{N}(\mu,\sigma)$	Log-normal distribution from $\mathcal{N}(\mu,\sigma^2)$
$H(\cdot;\mu,\sigma)$	Cumulative distribution function for $\mathcal{N}(\mu,\sigma^2)$
$h(\cdot;\mu,\sigma)$	Probability density function for $\mathcal{N}(\mu,\sigma^2)$
$W(\cdot)$	Lambert-W function
	Notation specific to Section 3
$h_n(\cdot)$	Probability mass function for Poisson distribution
$\breve{p}_n$	Maximum value attained by $h_n(\cdot)$
$\kappa_n(\cdot)$	Conversion function
	Notation specific to Section 4
$g(\cdot)$	Rate of reward

This page is intentionally blank

1. Introduction

There are military operations that can be modelled as a contest between a sensor that alternates between working and broken at random times and a target that reluctantly gives away glimpses. The *intermittent sensor homogenous glimpses model* goes further by modeling the target as giving away glimpses as a homogenous Poisson process. The model has been applied to uninhabited aerial vehicles hunting for time-sensitive targets [6] and submarines standing off from counter-detection [8].

In studies of such operations, the key measures of performance are based on the probability of detecting the target with the sensor over an interval of time. Specifically, it is useful to consider the probability of detecting n glimpses during an interval of time (especially the case of detecting $n = 0$ glimpses which equates to not detecting the target) as this measures the ability to obtain enough information about a target to take action against it. Indeed a single glimpse may be enough to infer that the target is nearby, but multiple glimpses may be needed if the target is to be classified, localized, and hence attacked. It is likewise useful to consider the probability of detecting the k-th glimpse during an interval of time and of detecting the k-th glimpse when there are n during that interval, as this measures the ability to collect a specific piece of intelligence from a target. For example, if the goal is to record the emissions from a target so that it can be recognized in the future, then it may be necessary to capture specific emissions within the total set that occurs over a time interval.

This report obtains approximations to those measures of performance that are easy to calculate, and proves that the approximations are close to reality when the time interval is long. So while the intermittent sensor homogenous glimpses model is evidently an abstraction of real life, it can be sufficiently valid for a first, ‘back of the envelope’ analysis. Moreover the approximations provide insight into how performance will behave overall, something that can be difficult to obtain from stochastic simulation only. Previous literature has considered *intermittent search* in which the searcher has phases of slow motion in which it can detect the target, and phases of fast motion when it cannot detect the target [3]. Intermittency can arise from many causes, for example: from motion of the sensor platform [9] (see discussion on ‘terrain masking’), obscurants in the environment [2] [12], or by deliberately turning the sensor off and on to avoid counter-acquisition [20]. The report diverges from research into *adaptive sensing* [5] in which sensor resources are actively managed to achieve a sensor task. In an adaptive sensing context, one might intermittently change the sensor from one mode to another so that the performance of the sensor is maximized. In this report, the sensor is intermittently alternating between working and broken states for reasons that are not controllable and have to be coped with.

The novel contribution of this report is in the calculation of closed-form approximations to the probabilities of detecting n glimpses during an interval of time, detecting the k-th glimpse during an interval of time, and detecting the k-th glimpse when there are n during that interval. The report is aimed at analysts who are contemplating the intermittent sensor homogenous glimpses model for their studies of operations. It provides a consolidated presentation of the model and calculations arising from it as a reference for future work. Section 2 works through the model and its validity. Section 3 considers the probability of detecting n glimpses during an interval of time, and in particular the case $n = 0$ (target not detected). Section 4 looks at the probability of detecting the k-th glimpse during an interval of time and of detecting the k-th glimpse when there are n during that interval. We conclude with advice on how analysts can apply the findings.

2. Intermittent Sensor Homogenous Glimpses

2.1. Model

The intermittent sensor homogenous glimpses model consists of a sensor that is working or broken under an alternating renewal process versus a target that reluctantly gives away glimpses as a homogenous Poisson process (Figure 1). In detail:

- The target gives away glimpses as a homogenous Poisson process with mean time between glimpses z. If a glimpse arrives when the sensor is working then the glimpse will be detected, but if the sensor is broken then the glimpse will be missed. We put $\zeta = \frac{1}{z}$ as the *mean glimpse rate* (mean rate at which the target gives away glimpses).

- At any point in the time interval $[0, \tau]$, the sensor is either *working* or *broken*. The working durations $W_1, W_2, \dots, W_k, ...$ are alternated with the broken durations $B_1, B_2, \dots, B_k, \dots$ where $\{(W_k, B_k) : k = 1, 2, \dots\}$ is a sequence of mutually independent, identically distributed, non-negative, vector random variables that are drawn from a non-degenerate bivariate distribution. Note that the *cycle durations* $\{W_k + B_k : k = 1, 2, \dots\}$ are mutually independent so we have an alternating renewal process. To be explicit, each broken duration B_k is allowed to depend on the working duration W_k for the k-th cycle.

- Let ρ denote the correlation of B_k with W_k (for all k as $\{(W_k, B_k)\}$ are identically distributed). Note that if the sequences $\{W_k\}$ and $\{B_k\}$ are independent then $\rho = 0$ (the converse is not necessarily true).

- We assume that the mean μ_{F} and variance σ_{F}^2 of the working durations are finite and positive. Likewise the mean μ_{R} and variance σ_{R}^2 of the broken durations are assumed to be finite and positive. These assumptions will be used in the proof of Lemma 1.

- The calculations that consider the k-th glimpse (Section 4) will further assume that $\mathbb{E}(W_k^3) < \infty$ and $\mathbb{E}(B_k^3) < \infty$; that is, the working and broken durations have finite skewness (in addition to having finite mean and variance). This assumption will be used in the proof of Lemma 2, which is used by the calculations about the k-th glimpse.

In any given application, it will be necessary to check that the model is valid abstraction of reality as opposed to being a 'strawman'. Working through the assumptions:

- That the sensor is either working or broken is a gross simplification of the real world. An accurate model would vary z with the distance of the sensor to the target, the target's susceptibility to detection (its signature), the environment, and other factors. The assumption of z constant (a homogenous Poisson process) can nonetheless provide a first-order insight into operations. The practical interest is in z large, namely a target

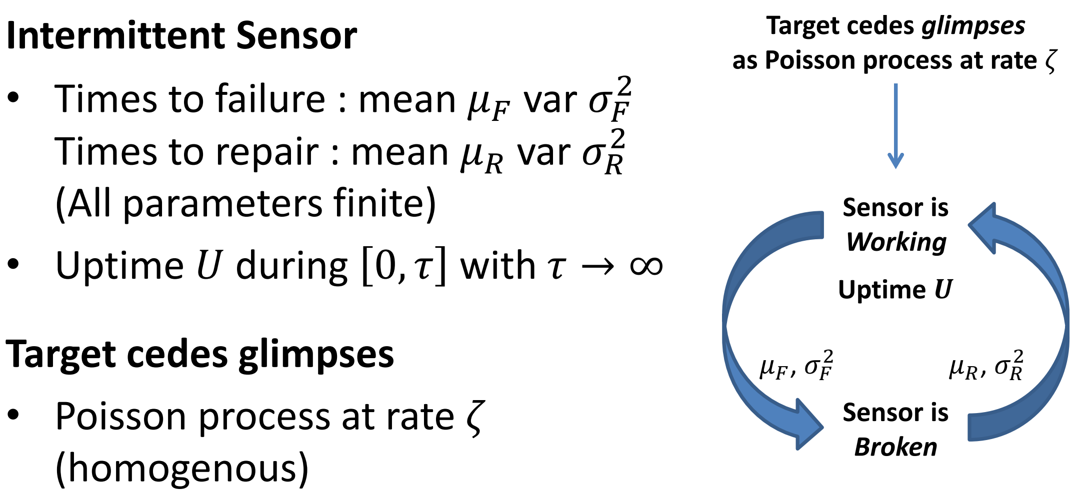

Figure 1: The intermittent sensor homogenous glimpses model.

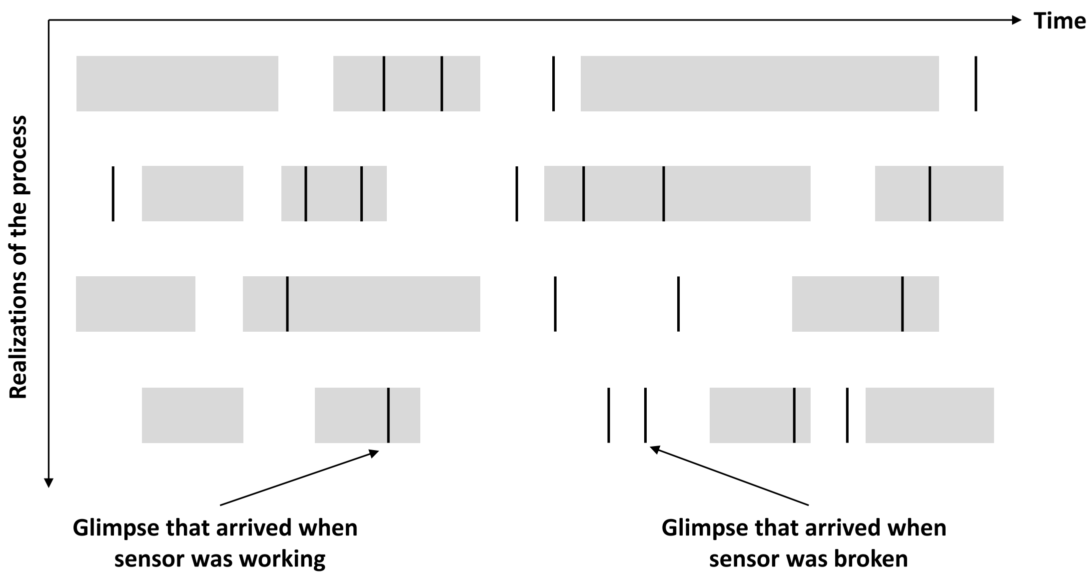

Figure 2: A realization of the intermittent sensor homogenous glimpses process consists of durations in which the sensor is alternately working (shown as shaded rectangles) and then broken (unshaded unrectangles), and a set of glimpses that are reluctantly given away by the target (marked as vertical line segments). If a glimpse arrives at a time when the sensor is working then it is detected. Otherwise if a glimpse arrives at a time when the sensor is broken then it is missed.

that is difficult to detect as it rarely gives away glimpses.

- Modeling targets as giving away glimpses as a Poisson process (possibly non-homogenous) follows common practice in studies of search and screening. Koopman [11, Section 3.2] discusses detection when looking continuously for a target, and calls ζ the 'instantaneous probability density of detection'. Wagner et al. [19, Section 507] references Koopman's work in the discussion of 'detection rate' modeling under assumptions of independent and continuous looking, with the homogenous Poisson process being referred to as the 'constant detection rate' model with parameter ζ.

- The model covers both passive and active sensors, in that it is only concerned with opportunities for the sensor to see the target. For passive sensors, the target might be continuously emitting a signal that is broken up stochastically by the environment, or randomly emitting bursts of detectable energy. Active sensors can be thought of as 'pinging' (illuminating) the environment with energy and listening for the backscatter. Again, the backscattered energy will be broken up stochastically by the environment. That said, if the 'pings' come at discrete times then it may be more appropriate to apply a discrete model of target detection; see [11, Section 3.2] and [19, Section 502–504].

- In assuming that the cycle durations are mutually independent, the sensor is implicitly assumed to 'reset' with each cycle. The assumption is reasonable in the absence of opposing arguments.

The model makes no assumptions about the sensor's state at time $t = 0$. Indeed it is well-known [18] that an alternating renewal process will forget the state that it was in at time $t = 0$, in that as time passes the probability of being in the working state approaches the *stationary probability* $\frac{\mu_{\mathrm{F}}}{\mu_{\mathrm{R}}+\mu_{\mathrm{F}}}$. For completeness, we note that an alternating renewal process is said to be *ordinary* if it the process is working at $t = 0$ versus *in equilibrium* if its probability of being in the working state at $t = 0$ is the stationary probability. In a real-world operation, it is arguably more realistic to assume that the sensor is working at the start of an operation. That said, one might assume that the sensor is allowed to 'run in' a bit and hence the process can be taken as being in equilibrium.

The analysis in this report only requires the means and variance of the durations spent working W_k and broken B_k, and does not require their actual distributions. But as an aid to future investigations, the following observations are made about potential distributions for W_k and B_k: For 'back of the envelope' work, one might assert that either or both of the durations follow exponential distributions as it has the well-known property of being memoryless. If the times to failure are based on failures in equipment or materials then one might adopt the exponential distribution [14], the Weibull distribution [10], or the Birnbaum-Saunders distribution [15, 8.1.6.6]. On the other hand, log-normal distributions may be justifiable given (for example) their utility in modeling the ability of sensors to see through the water [4]. Finally, uniform distributions or triangular distributions could be applied in situations where the sensor is exposed for a duration that has a fixed cut-off time.

2.2. Foundation calculations

Throughout this report, let $\mathbb{R}$ denote the real numbers, $\mathbb{P}(\cdot)$ be 'the probability of $\cdot$' and $\mathbb{E}(\cdot)$ be 'the expected value of $\cdot$'. Write $\mathcal{N}(\mu, \sigma^2)$ for the normal distribution with mean μ and variance σ^2.

Our interest is in glimpses that arrive when the sensor is working. Hence we focus on the sensor's *uptime* U during a given time interval $[0, \tau]$. Some classic results in reliability theory [16] [17] establish that under the assumptions made about the dependence between $\{W_k\}$ and $\{B_k\}$ (the durations that that the sensor is working and broken), the U is asymptotically normal — intuitively, U is approximately normal for any τ and the approximation becomes perfect as $\tau \to \infty$.

Lemma 1 (Uptime is asymptotically normal)**.** Let U be the *uptime*, namely the cumulative duration that the sensor is working during an interval $[0, \tau]$. Then as $\tau \to \infty$, the quantity $\dfrac{U - \mu_U}{\sigma_U}$ converges in distribution to $\mathcal{N}(0, 1)$ where

$$
\begin{aligned}
\mu_U &= c\tau \\
\sigma_U^2 &= 2c(1-c)w\tau \\
c &= \frac{\mu_{\mathrm{F}}}{\mu_{\mathrm{R}} + \mu_{\mathrm{F}}} \\
w &= \frac{\mu_{\mathrm{R}}^2 \sigma_{\mathrm{F}}^2 + \mu_{\mathrm{F}}^2 \sigma_{\mathrm{R}}^2 - 2\rho\mu_{\mathrm{R}}\mu_{\mathrm{F}}\sigma_{\mathrm{R}}\sigma_{\mathrm{F}}}{2\mu_{\mathrm{R}}\mu_{\mathrm{F}}(\mu_{\mathrm{R}} + \mu_{\mathrm{F}})}
\end{aligned}
$$

Proof. If the sequences $\{W_k\}$ and $\{B_k\}$ are independent then the result follows immediately from [16, Example 1]. Otherwise let $\mathcal{F}$ be the joint (bivariate) distribution for B_k with W_k (for all k as $\{(W_k, B_k)\}$ are identically distributed). Now $\mathcal{F}$ is non-degenerate by assumption and B_k and W_k have finite variance for all k. Therefore $\mathcal{F}$ is contained in the normal domain of attraction of a bivariate normal distribution [1] and the result follows by [17, Example 3]. □

It will also be useful to think of the sensor as accumulating a reward when it is working.

Lemma 2 (Accumulated reward is asymptotically normal)**.** Let X_t denote the sensor's state at time t wherein $X_t = 0$ if the sensor is broken and $X_t = 1$ if it is working. Given $g : [0, 1] \to \mathbb{R}$, put $Q = \int_0^\tau g(t/\tau) X_t \, dt$ (reward the sensor at rate $g(t/\tau)$ if it is working at time t), and set

$$
\begin{aligned}
\mu_Q &= \bar{g}\mu_U \\
\sigma_Q^2 &= \gamma\sigma_U^2
\end{aligned}
$$

where $\bar{g} = \int_0^1 g(x)\, dx$, $\gamma = \int_0^1 (g(x))^2\, dx$, and μ_U, σ_U^2 are provided by Lemma 1. Suppose that all of the following conditions are satisfied:

- $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty, \mathbb{E}(B_k^3) < \infty$, for all k.
- $-\infty < \bar{g} < \infty$, $0 < \gamma < \infty$, and $|\int_0^1 g(x)g'(x)\, dx| < \infty$.

Then as $\tau \to \infty$, the quantity $\dfrac{Q - \mu_Q}{\sigma_Q}$ converges in distribution to $\mathcal{N}(0, 1)$.

Proof. See [7]. □

Remark. To approximate U by a normal distribution we need only obtain the parameters μ_U, σ_U^2. Likewise to approximate Q by a normal distribution we need only obtain the parameters μ_Q, σ_Q^2 which are fully determined by the reward rate g and the values μ_U, σ_U^2. Moreover under our assumptions about the dependence between the durations spent working W_k versus broken B_k, the values μ_U, σ_U^2 are fully determined by the means and variances of those durations (μ_{F}, μ_{R}, σ_{F}^2, σ_{R}^2) and the coefficient ρ. All other information about the distributions of W_k or B_k is ignored.

The following lemma is for technical purposes. It establishes that for certain random variables, if we can approximate the distribution of the variable then we can use that approximation to estimate the variable's mean.

Lemma 3. Let $\{F_\tau\}_\tau$ and $\{G_\tau\}_\tau$ be sequences of cumulative distribution functions, and $\{X_{F_\tau}\}_\tau$ and $\{X_{G_\tau}\}_\tau$ be the corresponding sequences of random variables. Suppose that for all τ, X_{F_τ} and X_{G_τ} are both continuous and non-negative, and $\mathbb{E}(X_{F_\tau})$ and $\mathbb{E}(X_{G_\tau})$ are both finite. Suppose further that for all $x \geq 0$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|F_\tau(x) - G_\tau(x)| < \epsilon$. Then for all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.

Proof. See Appendix. □

3. Detecting n Glimpses

This section considers the probability P_n of detecting n glimpses during the time interval $[0, \tau]$. A practical application is in operations in which a searcher has to see a required number of glimpses before they can take some action. Consider, for example, an aircraft that is hunting for a surface-to-air missile battery by listening for its radar emissions. The radar emissions can be treated as opportunities to glimpse the battery. A single glimpse may be enough to infer that the battery is nearby, but multiple glimpses may be needed if the battery is to be classified, localized, and hence attacked. As notation for this section, let $\ln\mathcal{N}(\mu, \sigma)$ denote the log-normal distribution associated with $\mathcal{N}(\mu, \sigma^2)$. If $Y \sim \mathcal{N}(\mu, \sigma^2)$ then we write $\mathcal{N}(y; \mu, \sigma) = \mathbb{P}(Y \leq y)$. Similarly if $X \sim \ln\mathcal{N}(\mu, \sigma)$ then $\ln\mathcal{N}(y; \mu, \sigma) = \mathbb{P}(Y \leq y)$.

Lemma 1 established that the uptime U during time interval $[0, \tau]$ is asymptotically normal. Meanwhile, the target is giving away glimpses as a homogenous Poisson process at rate ζ. Hence the number of glimpses during the time interval will be a random variable that we can calculate as ζU *and* that random variable will also be asymptotically normal. We formalize this idea in the following two lemmas as the stepping stone to the main results for this section.

Lemma 4. $\left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| = \left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right|$.

Proof. See Appendix. □

Lemma 5 (Number of glimpses is asymptotically normal)**.** For any $u \in \mathbb{R}$, $\epsilon > 0$:

1. There exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(\zeta U \leq \zeta u) - \mathcal{N}(\zeta u; \zeta\mu_U, \zeta^2\sigma_U^2)\right|$.
2. There exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right|$.

Proof. Take Lemma 1 and unpack the definition of convergence in distribution: for any $u \in \mathbb{R}$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then

$$\left|\mathbb{P}\left(\frac{U - \mu_U}{\sigma_U} \leq \frac{u - \mu_U}{\sigma_U}\right) - \mathcal{N}(u; 0, 1)\right| < \epsilon$$

By properties of the normal distribution this holds if and only if

$$\left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| < \epsilon$$

Hence for result (1) we apply properties of the normal distribution to get

$$\left|\mathbb{P}(\zeta U \leq \zeta u) - \mathcal{N}(\zeta u; \zeta\mu_U, \zeta^2\sigma_U^2)\right| < \epsilon$$

Meanwhile for result (2) we apply Lemma 4 to get

$$\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right| < \epsilon$$

□

Remark. Lemma 5 establishes that $\mathbb{P}(\zeta U \leq \zeta u)$ can be approximated by a normal distribution, and that the approximation improves as $\tau \to \infty$, but it does *not* calculate a value of τ that will guarantee the goodness of the approximation. In technical terms, Lemma 5 establishes convergence but does not supply a rate of convergence. In particular, while Lemma 5 establishes that τ' exists for given $u \in \mathbb{R}$, $\epsilon > 0$, it does not calculate a value for it. At its heart, Lemma 5 calls on Lemma 1 to obtain τ'. So if a calculation for τ' is desired, it will be necessary to replace Lemma 1 with a stronger result.

3.1. Detecting Zero Glimpses

The following result deduces that the distribution of P_0 approaches log-normal as $\tau \to \infty$.

Proposition 1. Set $\mu = -\zeta\mu_U$, $\sigma^2 = \zeta^2\sigma_U^2$. For any $0 \leq p \leq 1$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(P_0 \leq p) - \ln\mathcal{N}(p; \mu, \sigma^2)\right| < \epsilon$.

Proof. Suppose $\epsilon > 0$. Given p, construct $u = -\frac{1}{\zeta}\ln(p)$ so $p = e^{-\zeta u}$. By Lemma 5 there exists $\tau' > 0$ such that if $\tau > \tau'$ then

$$\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; \mu, \sigma^2)\right| < \epsilon$$

Now $P_0 = \mathbb{P}(\text{Zero glimpses acquired during } [0, \tau]) = e^{-\zeta U}$. Moreover the function $\exp(\cdot)$ is strictly increasing so $-\zeta U \leq -\zeta u$ if and only if $P_0 \leq p$. Hence $\mathbb{P}(P_0 \leq p) = \mathbb{P}(-\zeta U \leq -\zeta u)$ so if $\tau > \tau'$ then

$$\left|\mathbb{P}(P_0 \leq p) - \mathcal{N}(\ln p; \mu, \sigma^2)\right| < \epsilon$$

or equivalently

$$|\mathbb{P}(P_0 \leq p) - \ln \mathcal{N}(p; \mu, \sigma)| < \epsilon$$

□

We immediately obtain an approximation to the expected value of P_0 that becomes perfect as $\tau \to \infty$.

Corollary 1. Put

$$\nu_\tau = \exp\left(\zeta c\tau(\zeta(1-c)w - 1)\right)$$

Then $|\mathbb{E}(P_0) - \nu_\tau| \to 0$ as $\tau \to \infty$ provided $\zeta \leq \hat{\zeta}$ where

$$\hat{\zeta} = \frac{1}{2(1-c)w}$$

Proof. Let $X \sim \ln \mathcal{N}(\mu, \sigma)$. Lemma 3 finds that $|\mathbb{E}(P_0) - \mathbb{E}(X)| \to 0$ as $\tau \to \infty$. We then use known properties of the log-normal distribution to calculate

$$\begin{aligned}\mathbb{E}(X) = e^{\mu+\sigma^2/2} &= \exp\left(-\zeta c\tau + \zeta^2 c(1-c)w\tau\right) \\ &= \exp\left(\zeta c\tau(\zeta(1-c)w - 1)\right)\end{aligned}$$

We now describe why the caveat $\zeta \leq \hat{\zeta}$ is required. Observe that $\exp(\cdot)$ is an increasing function and $f(\zeta) = \zeta c\tau(\zeta(1-c)w-1)$ is a positive quadratic with inflection point $\hat{\zeta}$. Hence f is decreasing on $\zeta \leq \hat{\zeta}$, but increasing thereafter. But in reality, we should have $\mathbb{E}(X) \to 0$ as $\zeta \to \infty$: if the target is giving away a huge number of glimpses then it is bound to be acquired by the sensor, whereby $P_0 \to 0$ surely and hence $X \to 0$ surely. Thus we use the caveat to constrain ζ to the domain on which f is decreasing. □

Remark. The approximation to $\mathbb{E}(P_0)$ provided by Corollary 1 is an easily-calculated measure of performance for the sensor. Indeed $1 - P_0$ is the probability of detecting the target.

The underlying issue that leads to the caveat is that we have taken limits in a non-commutative order. The quantity ζu corresponds physically to the number of glimpses given away by the target during u. As $\zeta \to \infty$, we should see $\zeta u \to \infty$ surely, but instead we are tied to a normal distribution with mean $\mu \to -\infty$ and deviation $\sigma \to \infty$. The correct treatment takes $\zeta \to \infty$ first, and then $\tau \to \infty$. The caveat is for mathematical correctness. The practical interest is in ζ small, wherein the target gives away glimpses rarely.

Figures 3–5 compare Proposition 1's approximation to the distribution of P_0 with empirical results from simulation. We inspect a range of values for τ to gain assurance that the approximation does indeed improve as $\tau \to \infty$. We consider three different cases of distributions for times to fail and repair to give some assurance that the results hold in all cases. Each

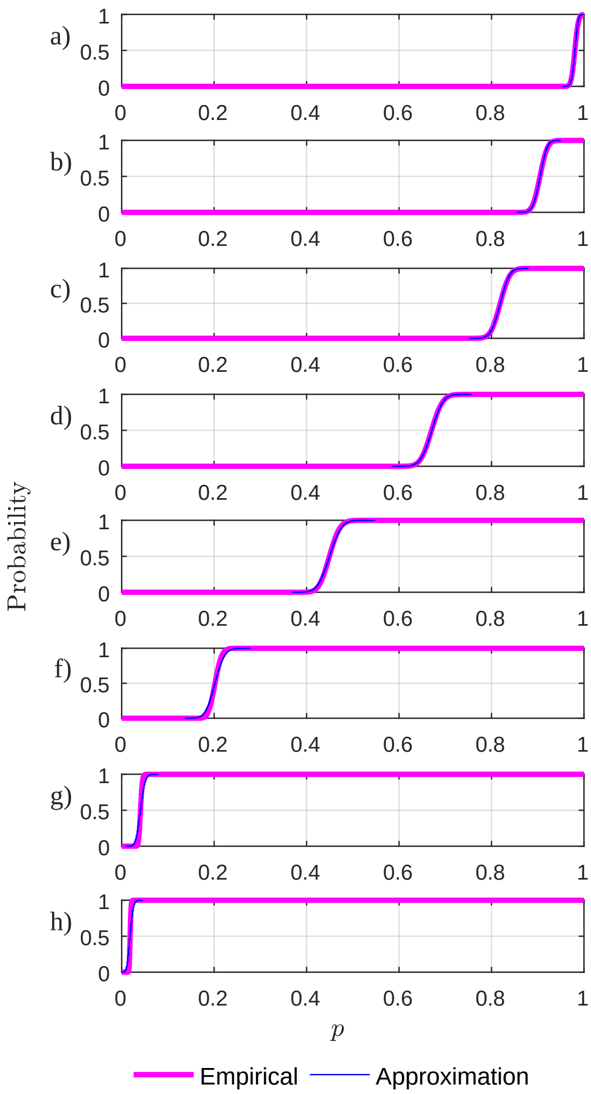

Figure 3: Approximation to distribution of P_0 *vs empirical distribution from simulations. Times to fail follow an exponential distribution with mean* 0.3*, times to repair follow an exponential distribution with mean* 0.7*,* $z = 150$*, a)* $\tau = 10$*. b)* $\tau = 50$*. c)* $\tau = 100$*. d)* $\tau = 200$*. e)* $\tau = 400$*. f)* $\tau = 800$*. g)* $\tau = 1600$*. h)* $\tau = 2000$*.*

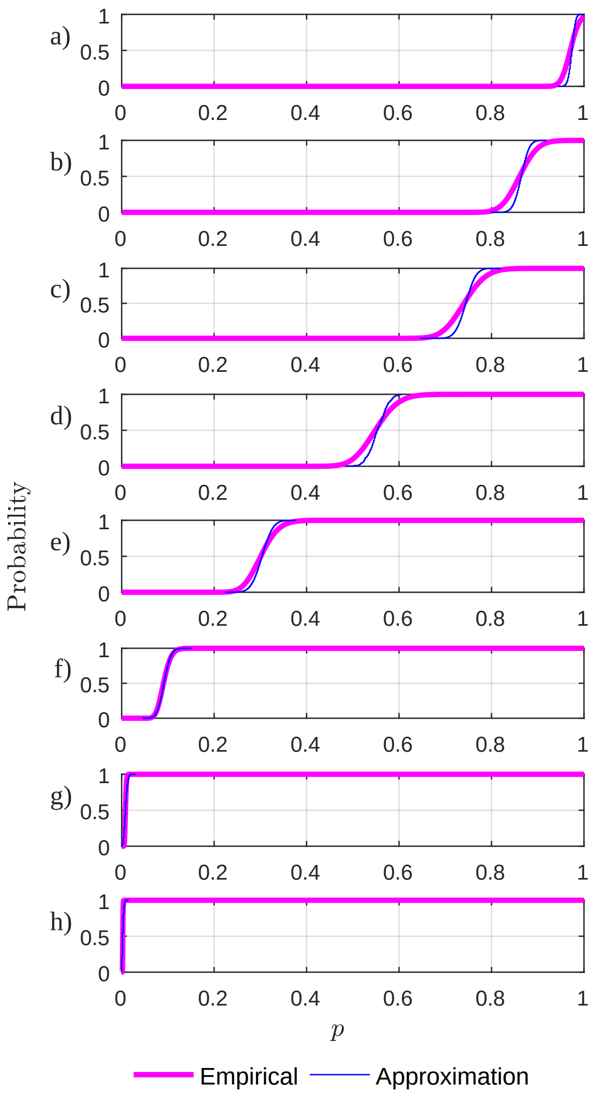

Figure 4: Approximation to distribution of P_0 *vs empirical distribution from simulations. Times to fail follow the log-normal distribution* $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*, a)* $\tau = 10$*. b)* $\tau = 50$*. c)* $\tau = 100$*. d)* $\tau = 200$*. e)* $\tau = 400$*. f)* $\tau = 800$*. g)* $\tau = 1600$*. h)* $\tau = 2000$*.*

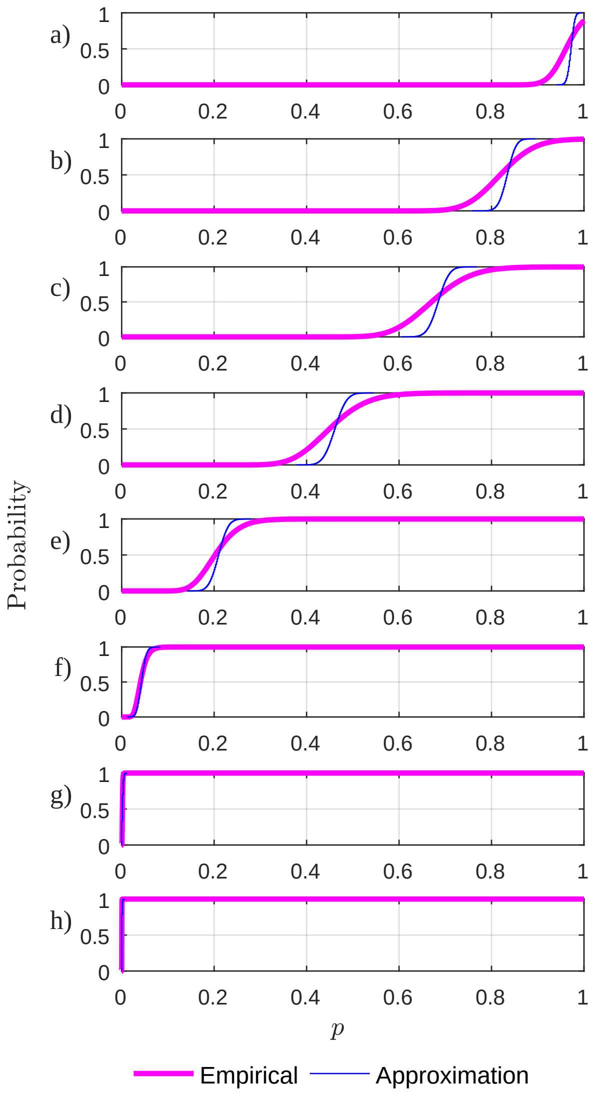

Figure 5: Approximation to distribution of P_0 vs empirical distribution from simulations. Times to fail follow the Birnbaum-Saunders distribution with $\beta = 3$, $\gamma = 3$, times to repair follow a triangular distribution on $[3, 8]$ with peak at 7, $z = 150$, a) $\tau = 10$. b) $\tau = 50$. c) $\tau = 100$. d) $\tau = 200$. e) $\tau = 400$. f) $\tau = 800$. g) $\tau = 1600$. h) $\tau = 2000$.

simulation run represented the sensor working intermittently during some interval $[0, \tau]$. The sensor had probability $\frac{\mu_F}{\mu_R + \mu_F}$ of being working at time $t = 0$ (simulation of the equilibrium process). A total of 4,000,000 runs were generated, and then P_0 was estimated 50,000 times. Each estimate used 600 of the runs, in sampling without replacement. Each figure shows the empirical cumulative distribution function from the 50,000 estimations, compared with the proposed approximation.

The distributions for P_0 match our intuitions: when $\tau = 0$ we have $P_0 = 0$ and as $\tau \to \infty$ we have $P_0 \to 1$ surely. While there is discrepancy between the approximate and empirical distributions, it disappears as $\tau \to \infty$.

3.2. Detecting One or More Glimpses

The following result establishes that when $n \geq 1$ the distribution of P_n can be thought of as approaching 'Lambert W-normal' as $\tau \to \infty$. For any positive integer n and $0 \leq p \leq 1$ put

$$\kappa_n(p) = -\frac{(n!p)^{1/n}}{n}$$

Let h be the probability mass function for the Poisson distribution, namely

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

It is readily shown that $h_n(\lambda)$ attains its maximum value of

$$\breve{p}_n = e^{-n}\frac{n^n}{n!}$$

when $\lambda = n$, and is increasing if $\lambda < n$ and decreasing if $\lambda > n$ (Appendix, Lemma 8). The function has two pre-images, namely

$$h^{-1}_{0,n}(p) = -nW_0(\kappa_n(p))$$
$$h^{-1}_{-1,n}(p) = -nW_{-1}(\kappa_n(p))$$

respectively mapping from $[0, \breve{p}_n]$ to $[0, n]$ and from $[0, \breve{p}_n]$ to $[n, \infty)$, where W_0, W_{-1} are the Lambert W function on its 0, -1 branches (Appendix, Lemma 9).

Proposition 2. Set $\mu = -\zeta\mu_U$, $\sigma^2 = \zeta^2\sigma_U^2$ and

$$F_\tau(p) = 1 - \mathcal{N}\big(W_0(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big) + \mathcal{N}\big(W_{-1}(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big)$$

For any $0 \leq p \leq 1$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{P}(P_n \leq p) - F_\tau(p)| < \epsilon$.

Proof. Suppose $\epsilon > 0$. Given p, construct $u_0 = \frac{1}{\zeta}h^{-1}_{0,n}(p)$, $u_{-1} = \frac{1}{\zeta}h^{-1}_{-1,n}(p)$ so $p = h_n(\zeta u_0)$ and $p = h_n(\zeta u_{-1})$. By Lemma 5 there exists $\tau'_0 > 0$ such that if $\tau > \tau'_0$ then

$$\big|\mathbb{P}(\zeta U \leq \zeta u_0) - \mathcal{N}(\zeta u_0; \zeta\mu_U, \zeta^2\sigma_U^2)\big| < \tfrac{1}{2}\epsilon$$

and likewise there exists $\tau'_{-1} > 0$ such that if $\tau > \tau'_{-1}$ then

$$\big|\mathbb{P}(\zeta U \leq \zeta u_{-1}) - \mathcal{N}(\zeta u_{-1}; \zeta\mu_U, \zeta^2\sigma_U^2)\big| < \tfrac{1}{2}\epsilon$$

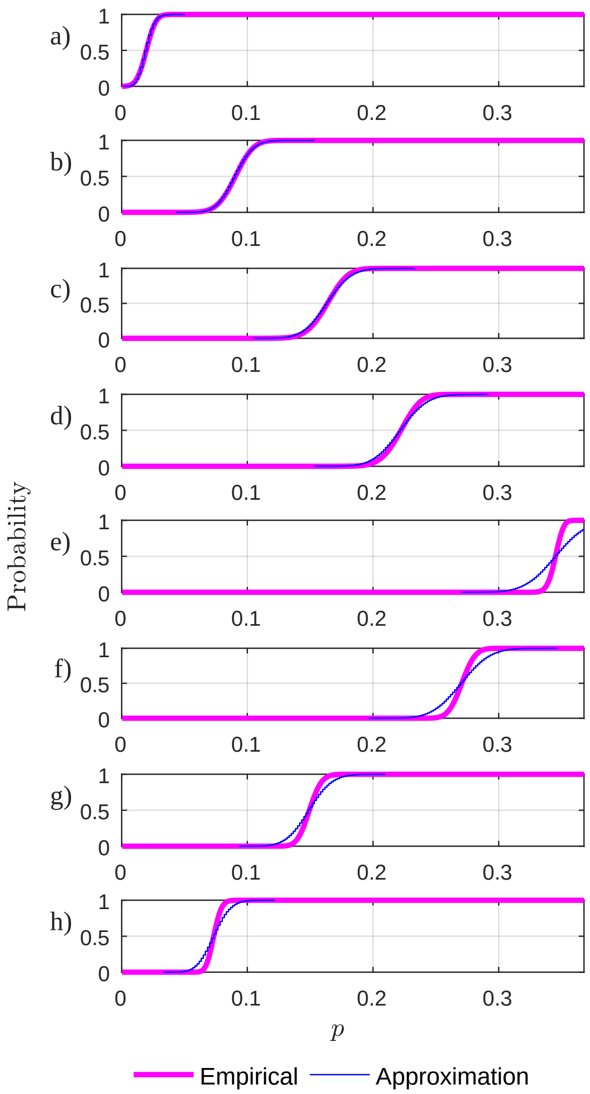

Figure 6: Approximation to distribution of P_1 *vs empirical distribution from simulations. Times to fail follow an exponential distribution with mean* 0.3*, times to repair follow an exponential distribution with mean* 0.7*,* $z = 150$*, a)* $\tau = 10$*. b)* $\tau = 50$*. c)* $\tau = 100$*. d)* $\tau = 150$*. e)* $\tau = 700$*. f)* $\tau = 1000$*. g)* $\tau = 1500$*. h)* $\tau = 2000$*.*

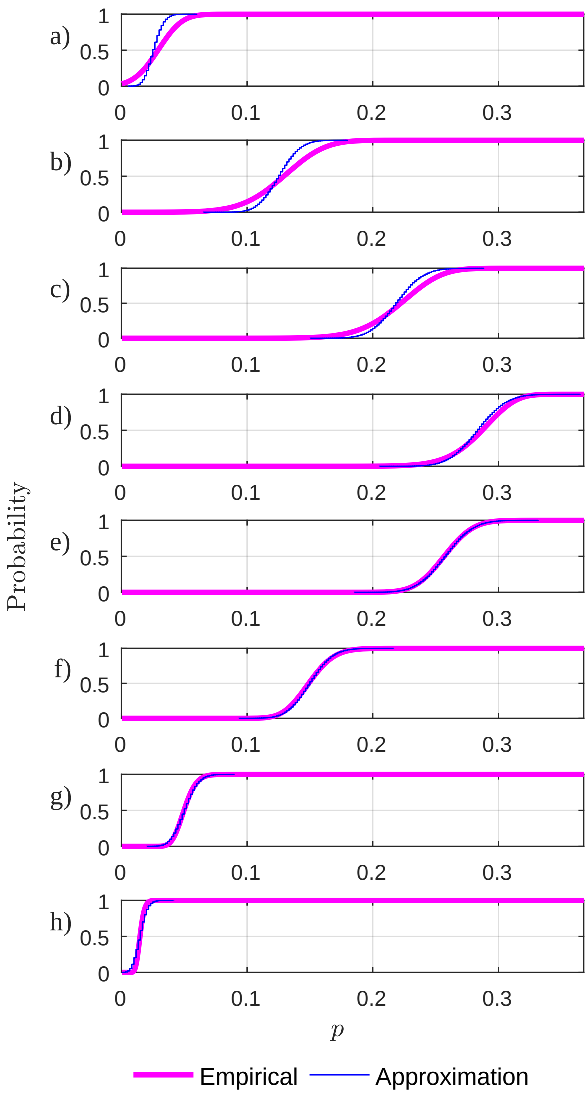

Figure 7: Approximation to distribution of P_1 vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*, a)* $\tau = 10$*. b)* $\tau = 50$*. c)* $\tau = 100$*. d)* $\tau = 150$*. e)* $\tau = 700$*. f)* $\tau = 1000$*. g)* $\tau = 1500$*. h)* $\tau = 2000$*.*

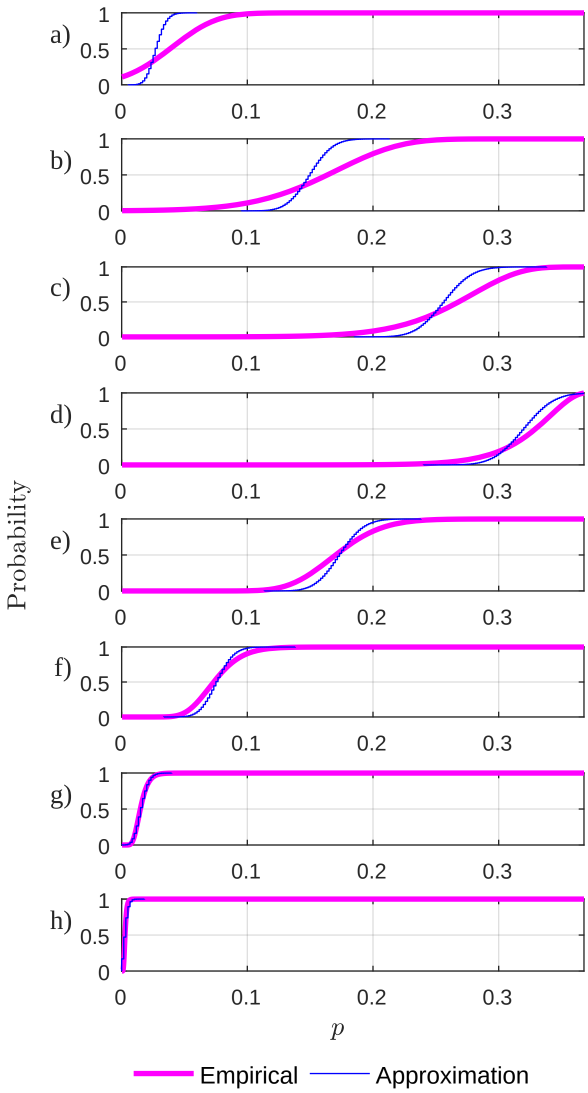

Figure 8: Approximation to distribution of P_1 *vs empirical distribution from simulations. Times to fail follow the Birnbaum-Saunders distribution with* $\beta = 3$, $\gamma = 3$, *times to repair follow a triangular distribution on* $[3, 8]$ *with peak at* 7, $z = 150$, *a)* $\tau = 10$. *b)* $\tau = 50$. *c)* $\tau = 100$. *d)* $\tau = 150$. *e)* $\tau = 700$. *f)* $\tau = 1000$. *g)* $\tau = 1500$. *h)* $\tau = 2000$.

Put $\tau' = \max(\tau'_0, \tau'_{-1})$. Now $P_n = \mathbb{P}(n \text{ glimpses acquired during } [0,\tau]) = h_n(\zeta U)$. Moreover $h_n(\cdot)$ is increasing on $[0, \breve{p}_n]$ and decreasing on $[\breve{p}_n, \infty)$ so $P_n \le p$ if and only if $\zeta U \le \zeta u_0$ or $\zeta U \ge \zeta u_{-1}$. Hence $\mathbb{P}(P_n \le p) = \mathbb{P}(\zeta U \le \zeta u_0) + \mathbb{P}(\zeta U \ge \zeta u_{-1})$ so if $\tau > \tau'$ then

$$\left|\mathbb{P}(P_n \le p) - (\mathcal{N}(h_{0,n}^{-1}(p); \zeta\mu_U, \zeta^2\sigma_U^2) + 1 - \mathcal{N}(h_{-1,n}^{-1}(p); \zeta\mu_U, \zeta^2\sigma_U^2))\right| < \epsilon$$

or equivalently

$$\left|\mathbb{P}(P_n \le p) - \left(1 - \mathcal{N}\big(W_0(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big) + \mathcal{N}\big(W_{-1}(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big)\right)\right| < \epsilon$$

□

As before, we get an approximation to the expected value of $\breve{p}_n$ that becomes perfect as $\tau \to \infty$.

Corollary 2. Put

$$\nu_\tau = \breve{p}_n - \int_0^{\breve{p}_n} F_\tau(p)\, dp$$

Then $|\mathbb{E}(P_n) - \nu_\tau| \to 0$ as $\tau \to \infty$.

Proof. Let $X \sim F_\tau$. Lemma 3 finds that $|\mathbb{E}(P_n) - \mathbb{E}(X)| \to 0$ as $\tau \to \infty$. Now it is well-known that $\mathbb{E}(X) = \int_0^\infty (1 - F_\tau(p))\, dp$ and the domain of F_τ is $[0, \breve{p}_n]$. □

Figures 6–8 compare Proposition 2's approximation to the distribution of P_n with empirical results from simulation. The simulations were conducted as for the $n = 0$ case. The distributions for P_n match our intuitions: when $\tau = 0$ we have $P_n = 0$. As τ increases, P_n initially concentrates around the value $\breve{p}_n$, but then as $\tau \to \infty$ we have $P_n \to 1$ surely. The discrepancy between the approximate and empirical distributions disappears as $\tau \to \infty$.

It is worth noting that when P_n is concentrated around $\breve{p}_n$, the proposed approximation to the distribution of P_n can take a long time to evaluate. The reason is that if $p \approx \breve{p}_n$ then $\kappa_n(p) \approx -\frac{1}{e}$ but naive implementations of W can take a long time to converge to accurate answers in this neighbourhood [13]. The issue can be addressed by using a careful implementation of W.

4. Detecting the k-th Glimpse

This section considers the probability P^k of detecting the k-th glimpse during the time interval $[0, \tau]$ and the probability P_n^k of detecting the k-th glimpse given that the target gives away n glimpses during that time interval. A practical application is in operations in which a specific piece of intelligence is sought from a target. Consider, for example, an aircraft that is trying to record the emissions from a surface-to-air missile radar so that in the future, if other aircraft hear those emissions then they know to evade. To sufficiently characterize the radar, it may be necessary to capture specific emissions within the total set that occurs over a time interval.

Our analysis hinges on the following lemma.

Lemma 6 (Probability of seeing a given glimpse is asymptotically normal)**.** Suppose that the waiting time to a glimpse has probability density function f. Let $Q \equiv Q(\tau)$ be the probability of seeing that glimpse during time interval $[0, \tau]$. Set

$$\mu_Q = \bar{g}\mu_U$$
$$\sigma_Q^2 = \gamma\sigma_U^2$$

where $g(s) = f(s\tau)$ for all $s \in [0,1]$, $\bar{g} = \int_0^1 g(x)\,dx$, $\gamma = \int_0^1 (g(x))^2\,dx$, and μ_U, σ_U^2 are provided by Lemma 1. Suppose that all of the following conditions are satisfied:

- $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty, \mathbb{E}(B_k^3) < \infty$, for all k.
- $-\infty < \bar{g} < \infty$, $0 < \gamma < \infty$, and $|\int_0^1 g(x)g'(x)\,dx| < \infty$.

Then as $\tau \to \infty$, the quantity $\dfrac{Q - \mu_Q}{\sigma_Q}$ converges in distribution to $\mathcal{N}(0,1)$.

Proof. During any infinitesimal interval $[t, t+\delta t]$, the glimpse will be detected at probability $f(t) \cdot \delta t$ if the sensor is working at time t. Hence the probability of seeing the glimpse is the accumulation of those probabilities over the full interval $[0, \tau]$. Algebraically, let X_t denote the sensor's state at time t wherein $X_t = 0$ if the sensor is broken and $X_t = 1$ if it is working. Then $Q = \int_0^\tau f(t)X_t\,dt = \int_0^\tau g(t/\tau)X_t\,dt$. Result follows from applying Lemma 2. □

Remark. Lemma 6 applies to any arrival process, not just the homogenous Poisson one. Lemma 6 establishes that $\mathbb{P}(\zeta U \leq \zeta u)$ the distribution of $(Q - \mu_Q)/\sigma_Q$ can be approximated by a $\mathcal{N}(0,1)$,and that the approximation improves as $\tau \to \infty$, but it does *not* calculate a value of τ that will guarantee the goodness of the approximation. In technical terms, Lemma 5 establishes establishes that $(Q - \mu_Q)/\sigma_Q$ converges in distribution to $\mathcal{N}(0,1)$, but does not supply a rate of convergence. If a rate of convergence is desired then it will be necessary to replace Lemma 2 with a stronger result.

4.1. Detecting the k-th Glimpse

We can immediately deduce an approximation to P^k.

Proposition 3. To obtain an approximation to P^k, apply Lemma 2 with $g(s) = f(s\tau; k, \zeta)$ where f is the probability density function for the Erlang distribution with rate parameter λ

$$f(t; k, \lambda) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \quad \text{where } t, \lambda \geq 0$$

Proof. The Erlang distribution provides the waiting time to the k-th arrival in a homogeneous Poisson process on $[0, \infty)$. Result follows from Lemma 6. □

Figures 9–11 compare Proposition 3's approximation to the distribution of P^k with empirical results from simulation. The simulations were conducted as for the previous section. The

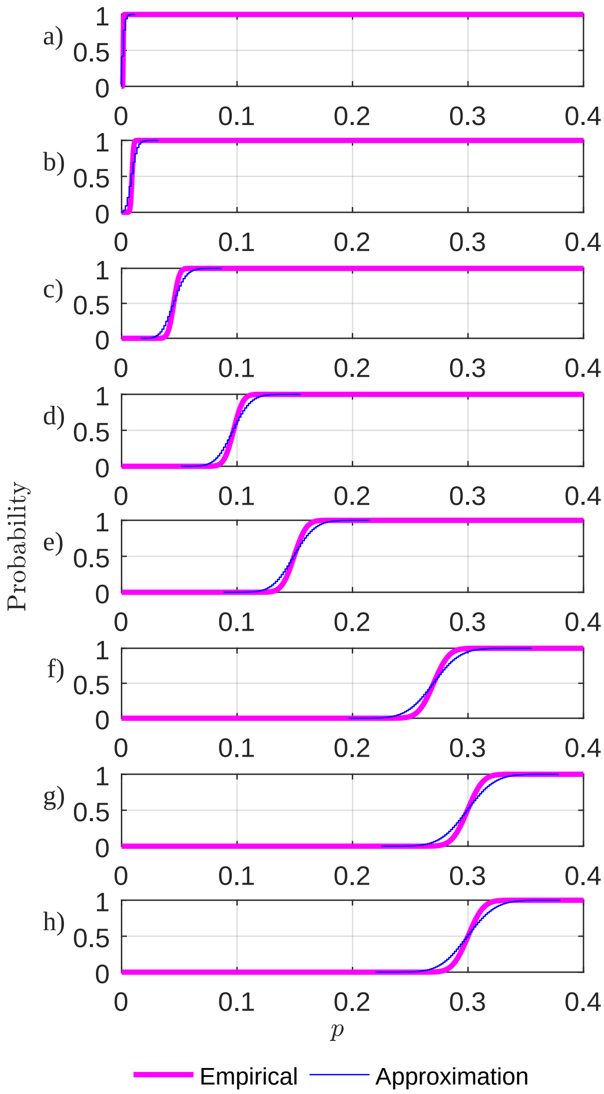

Figure 9: Approximation to distribution of P^3 *vs empirical distribution from simulations. Times to fail follow an exponential distribution with mean* 0.3*, times to repair follow an exponential distribution with mean* 0.7*,* $z = 150$*, a)* $\tau = 50$*. b)* $\tau = 100$*. c)* $\tau = 200$*. d)* $\tau = 300$*. e)* $\tau = 400$*. f)* $\tau = 800$*. g)* $\tau = 1600$*. h)* $\tau = 6400$*.*

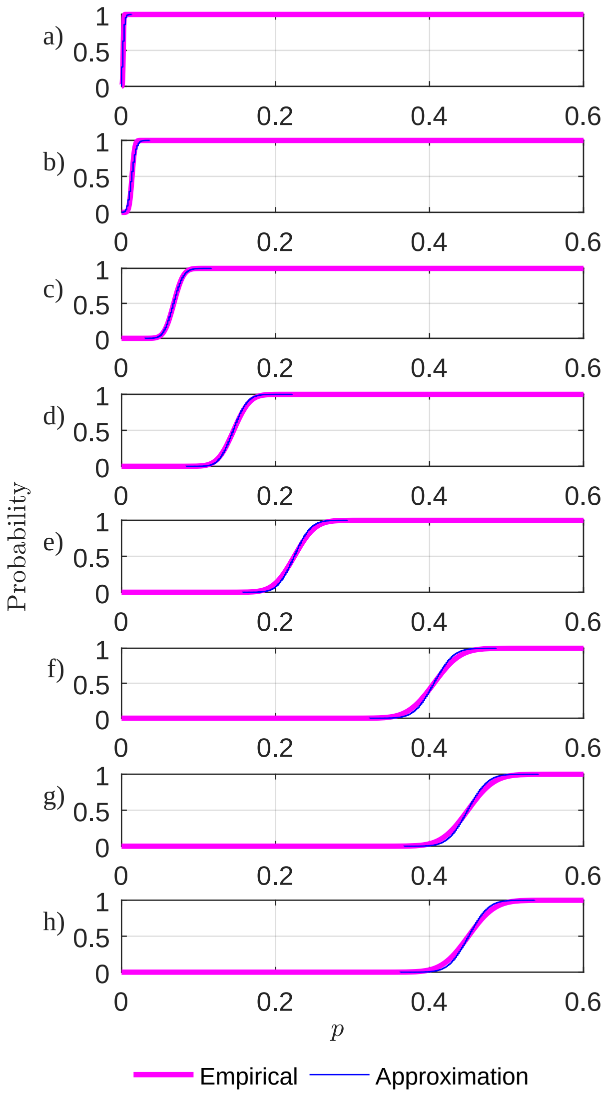

Figure 10: Approximation to distribution of P^3 *vs empirical distribution from simulations. Times to fail follow* $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*, a)* $\tau = 50$*. b)* $\tau = 100$*. c)* $\tau = 200$*. d)* $\tau = 300$*. e)* $\tau = 400$*. f)* $\tau = 800$*. g)* $\tau = 1600$*. h)* $\tau = 6400$*.*

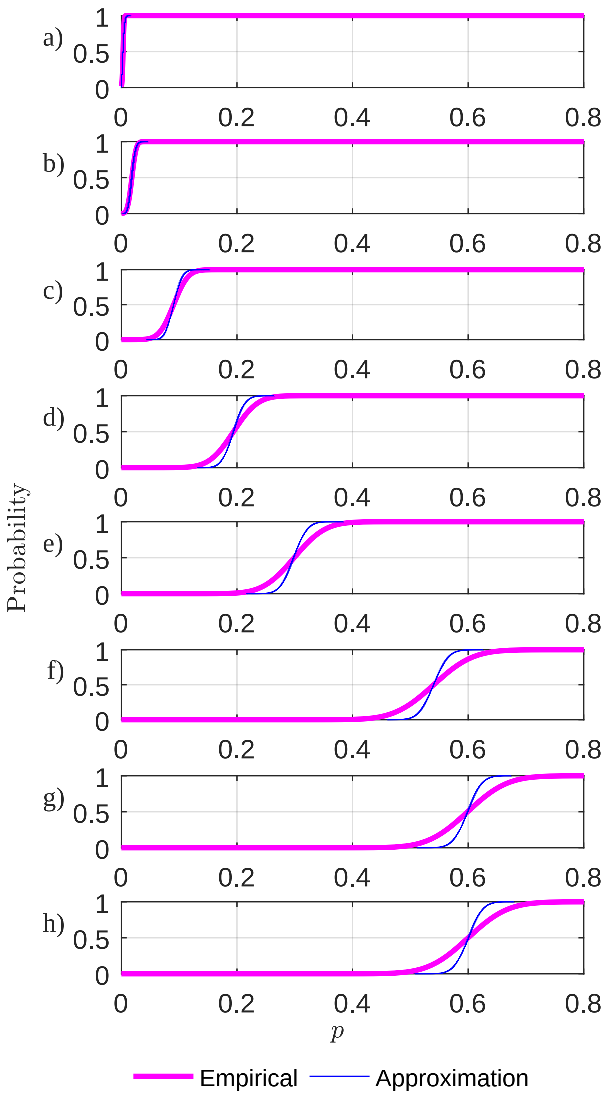

Figure 11: Approximation to distribution of P^3 vs empirical distribution from simulations. Times to fail follow the Birnbaum-Saunders distribution with $\beta = 3$, $\gamma = 3$, times to repair follow a triangular distribution on $[3, 8]$ with peak at 7, $z = 150$, a) $\tau = 50$. b) $\tau = 100$. c) $\tau = 200$. d) $\tau = 300$. e) $\tau = 400$. f) $\tau = 800$. g) $\tau = 1600$. h) $\tau = 6400$.

discrepancy between the approximate and empirical distributions disappears as $\tau \to \infty$. Intuitively, and as seen in the empirical distribution, we need τ large enough to have a chance of seeing the k-th glimpse; indeed $f(t; k, \zeta)$ attains its maximum at $t = (k-1)/\zeta = (k-1)z$ (Appendix, Lemma 10). Increasing τ beyond this value will not affect the probability – for a greater chance of seeing the glimpse, the sensor needs to be working more during $[0, \tau]$.

4.2. Detecting the k-th Glimpse Given n Glimpses

We now deduce an approximation to P_n^k. The result follows from the following lemma.

Lemma 7. The function

$$f_n(t; k, \zeta, \tau) = \frac{n!}{(k-1)!\,(n-k)!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^{n-k}$$

is the probability density function for the waiting time to the k-th arrival in a Poisson process on $[0, \infty)$ with rate parameter ζ given n glimpses during the time interval $[0, \tau]$.

Proof. Recall that the target is giving away glimpses as a homogeneous Poisson process with rate parameter ζ and let T_k be the time to the k-th arrival. Then T_k follows an Erlang distribution with rate parameter ζ so

$$\mathbb{P}(t \le T_k \le t + \delta t) = \frac{\zeta^k t^{k-1} e^{-\zeta t}}{(k-1)!} \cdot \delta t$$

But to see the k-th glimpse during $[0, \tau]$ there must be at least k glimpses during that interval. Thus we may also write

$$\begin{aligned}
\mathbb{P}(t \le T_k \le t + \delta t) &= \sum_{n=k}^{\infty} \mathbb{P}(t \le T_k \le t + \delta t | n \text{ glimpses in } [0, \tau]) \cdot \mathbb{P}(n \text{ glimpses in } [0, \tau]) \\
&= \sum_{n=k}^{\infty} \Big(f_n(t) \cdot \delta t\Big) \left(\frac{\lambda^n e^{-\lambda}}{n!}\right)
\end{aligned}$$

where $\sum_{n=k}^{\infty}$ means the infinite series as a sequence of partial sums, $f_n(t) \equiv f_n(t; k, \zeta, \tau)$, and we use $\lambda = \zeta\tau$ as the parameter to a Poisson distribution. Thus

$$\begin{aligned}
\frac{\zeta^k t^{k-1} e^{-\zeta t}}{(k-1)!} &= \sum_{n=k}^{\infty} f_n(t) \frac{\lambda^n e^{-\lambda}}{n!} \\
e^{\lambda - \zeta t} &= \sum_{n=k}^{\infty} f_n(t) \frac{(k-1)!}{\zeta^k t^{k-1}} \frac{\lambda^n}{n!} \\
&= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!}{\zeta^k t^{k-1}} \frac{\lambda^{n+k}}{(n+k)!} \\
&= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!}{t^{k-1}} \frac{\lambda^k}{\zeta^k} \frac{n!}{(n+k)!} \frac{\lambda^n}{n!} \\
e^{\lambda(1 - \frac{t}{\tau})} &= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!\,n!}{(n+k)!} \frac{\tau^k}{t^{k-1}} \frac{\lambda^n}{n!}
\end{aligned}$$

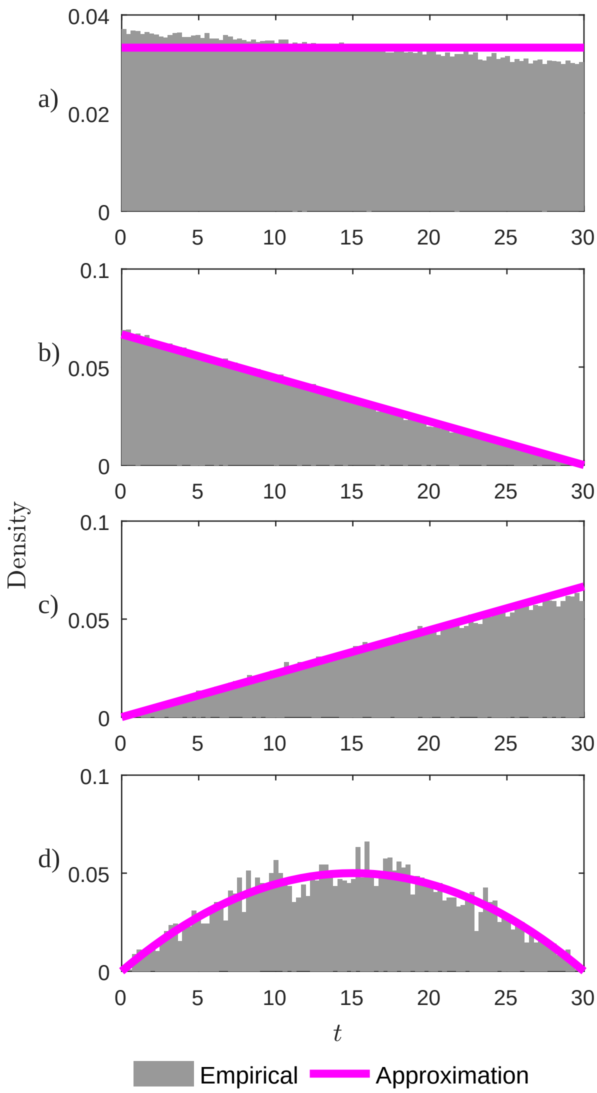

Figure 12: Predicted density of times to k-th glimpse given n glimpses during $[0, \tau]$ *vs empirical distribution from simulations.* $z = 150$, $\tau = 30$ *a)* $k = 1, n = 1$. *b)* $k = 1, n = 2$. *c)* $k = 2, n = 2$. *d)* $k = 2, n = 3$.

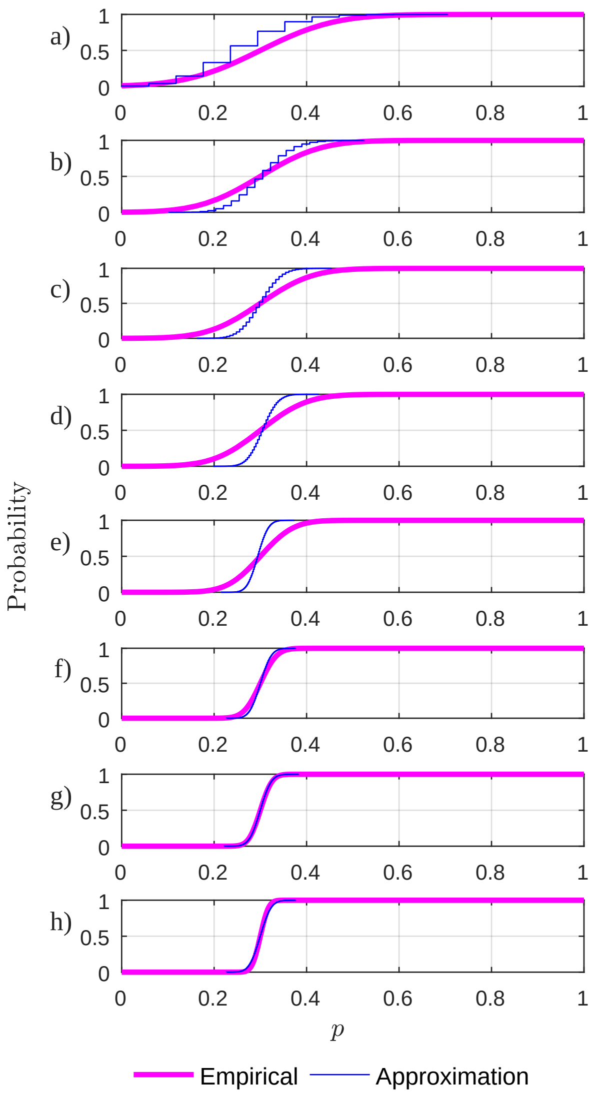

Figure 13: Approximation to distribution of P_3^1 vs empirical distribution from simulations. Times to fail follow an exponential distribution with mean 0.3, times to repair follow an exponential distribution with mean 0.7, $z = 150$, a) $\tau = 10$. b) $\tau = 15$. c) $\tau = 20$. d) $\tau = 25$. e) $\tau = 50$. f) $\tau = 200$. g) $\tau = 400$. h) $\tau = 800$.

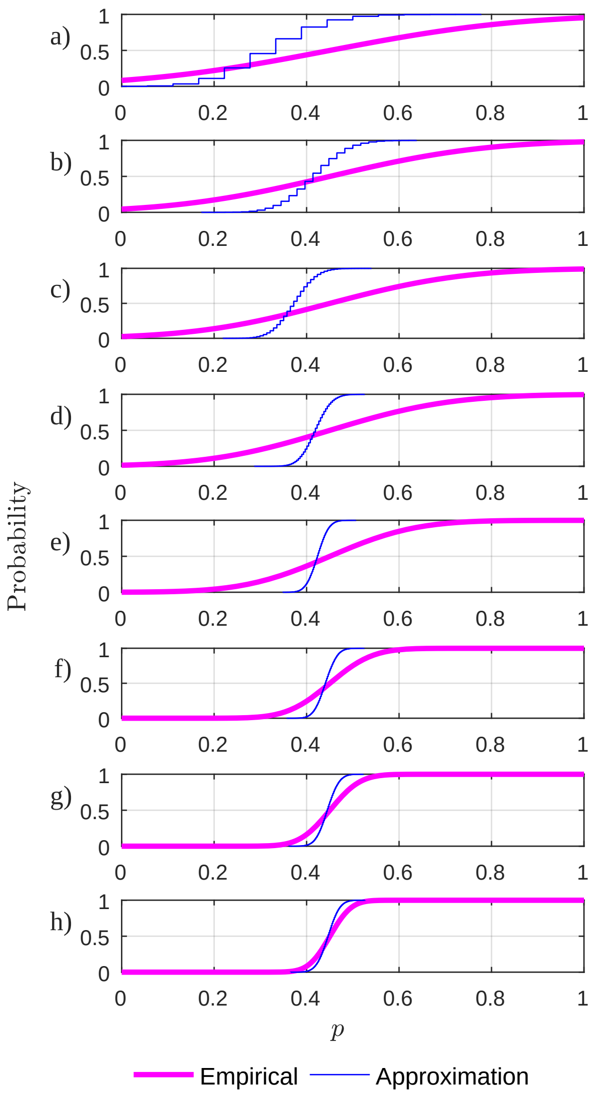

Figure 14: Approximation to distribution of P_3^1 vs empirical distribution from simulations. Times to fail follow $\ln\mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, a) $\tau = 10$. b) $\tau = 15$. c) $\tau = 20$. d) $\tau = 25$. e) $\tau = 50$. f) $\tau = 200$. g) $\tau = 400$. h) $\tau = 800$.

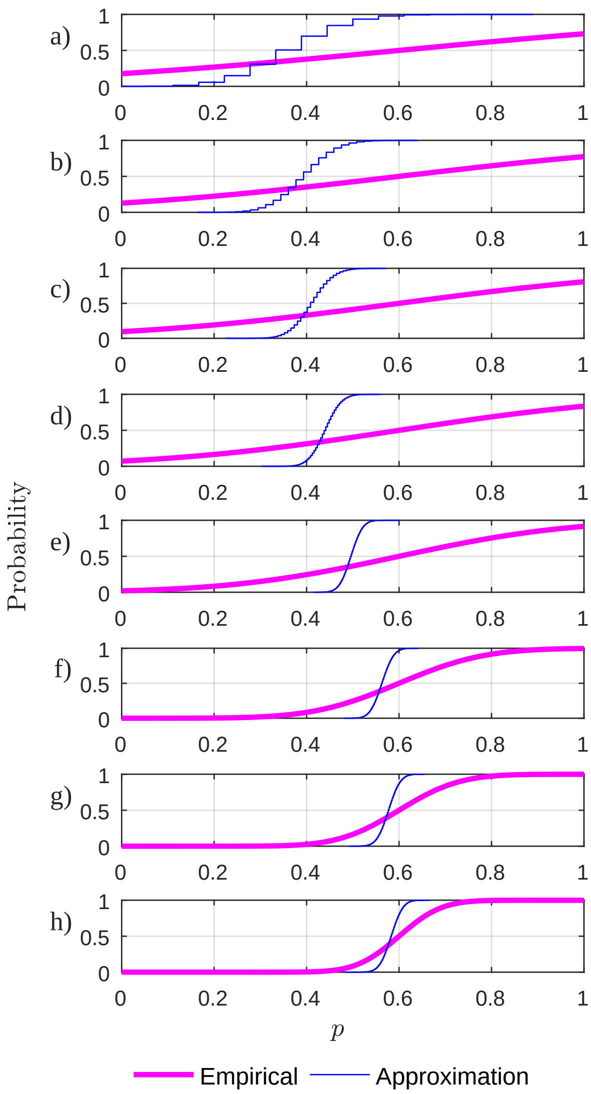

Figure 15: Approximation to distribution of P_3^1 *vs empirical distribution from simulations. Times to fail follow the Birnbaum-Saunders distribution with* $\beta = 3$, $\gamma = 3$, *times to repair follow a triangular distribution on* $[3, 8]$ *with peak at* 7, $z = 150$, *a)* $\tau = 10$. *b)* $\tau = 15$. *c)* $\tau = 20$. *d)* $\tau = 25$. *e)* $\tau = 50$. *f)* $\tau = 200$. *g)* $\tau = 400$. *h)* $\tau = 800$.

Now by the well-known expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ we have

$$e^{\lambda(1-\frac{t}{\tau})} = \sum_{n=0}^{\infty} \left(1 - \frac{t}{\tau}\right)^n \frac{\lambda^n}{n!}$$

where again $\sum_{n=0}^{\infty}$ means the infinite series as a sequence of partial sums. The two series are equal if and only if their terms are equal. Hence for all positive integers n we have

$$\begin{aligned} f_{n+k}(t) \frac{(k-1)!\, n!}{(n+k)!} \frac{\tau^k}{t^{k-1}} &= \left(1 - \frac{t}{\tau}\right)^n \\ f_{n+k}(t) &= \frac{(n+k)!}{(k-1)!\, n!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^n \\ f_n(t) &= \frac{n!}{(k-1)!\,(n-k)!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^{n-k} \end{aligned}$$

□

Proposition 4. To obtain an approximation to P_n^k, apply Lemma 2 with $g(s) = f_n(s\tau; k, \zeta, \tau)$.

Proof. Immediate from Lemma 6. □

Figure 12 checks that the waiting time to the k-th glimpse given n glimpses is indeed distributed in the manner predicted by Lemma 7. Figures 13–15 compare Proposition 4's approximation to the distribution of P_n^k with empirical results from simulation. The simulations were conducted as for the previous section. The agreement between the approximate and empirical distributions is poor when τ is small but improves as $\tau \to \infty$.

5. Conclusion

This report has studied a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time $[0, \tau]$, the sensor has probability P_n of detecting n glimpses, probability P^k of detecting the k-th glimpse, and probability P_n^k of detecting the k-th glimpse given n glimpses in that interval. The probabilities can provide insight into operations; indeed $1 - P_0$ is the probability of detecting the target, P_n considers the need to see the target multiple times, and P^k and P_n^k could apply in operations that seek to gather a targeted piece of intelligence from an adversary asset.

The research devised closed-form approximations to the distributions of P_n, P^k, P_n^k and proved that the approximations become perfect as $\tau \to \infty$ where 'perfect' is formally interpreted as pointwise convergence. Simulations of the equilibrium process showed that for τ finite, the approximations' closeness to reality can range from being poor to good (we did not obtain a guarantee on the rate at which the error decreases as τ increases).

The results can be applied to analysis of operations whenever the intermittent sensor homogenous glimpses model is a valid abstraction of the operation being studied. The means of

P_n, P^k, P_n^k could be useful as measures of performance as they can be easily estimated from the approximations that were obtained for those probabilities while being close to the actual values. Moreover we have learned that the distributions of P_n, P^k, P_n^k are each asymptotically 'function-normal', with a normal distribution around some function; indeed P_n is asymptotically log-normal when $n = 0$ and 'Lambert W-normal' when $n \geq 0$, and P^k and P_n^k are both asymptotically normal. This insight can guide the design and verification of simulations when studying the actual distributions of those probabilities.

6. Acknowledgements

The author thanks Stephen Bocquet for his feedback.

7. References

[1] Attraction domain of a stable distribution (2019) *Encyclopedia of Mathematics*, http://www.encyclopediaofmath.org/index.php?title=Attraction_domain_of_a_stable_distribution&oldid=43641.

[2] Babaria, B. R., Alvarez, T. L., Bergen, M. T. & Servatius, R. J. (2004) Transmission of light in a synthetic fog medium, *in IEEE 30th Annual Northeast Bioengineering Conference, 2004. Proceedings of the*, pp. 23–24.

[3] Bénichou, O., Loverdo, C., Moreau, M. & Voituriez, R. (2011) Intermittent search strategies, *Rev. Mod. Phys.* **83**, 81–129.

[4] Campbell, J. W. (1995) The lognormal distribution as a model for bio-optical variability in the sea, *Journal of Geophysical Research: Oceans* **100**(C7), 13237–13254.

[5] Chong, E. K. P., Kreucher, C. M. & Hero, A. O. (2008) Monte-Carlo-based partially observable Markov decision process approximations for adaptive sensing, *in 2008 9th International Workshop on Discrete Event Systems*, pp. 173–180.

[6] Hew, P. C. (2016) Processing, exploitation and dissemination: When is aided/automated target recognition "good enough" for operational use?, *in Operations Research for Unmanned Systems*, John Wiley & Sons, Ltd, pp. 139–153.

[7] Hew, P. C. (2017) Asymptotic distribution of rewards accumulated by alternating renewal processes, *Statistics & Probability Letters* **129**, 355 – 359.

[8] Hew, P. C. (2017) Cueing a submarine from standoff to ambush a target in an anti-submarine environment: How often can the cue be a false alarm?, *Journal of the Operational Research Society*.

[9] Hewitt, W. (1993) *Planting the Seeds of SEAD: The Wild Weasel in Vietnam*, AD-a425 679, Air University Press.

[10] Jiang, R. & Murthy, D. (2011) A study of weibull shape parameter: Properties and significance, *Reliability Engineering & System Safety* **96**(12), 1619 – 1626.

[11] Koopman, B. (1946) *Search and Screening*, OEG report, Operations Evaluation Group, Office of the Chief of Naval Operations, Navy Department.

[12] Li, Z., Nadon, S. & Cihlar, J. (2000) Satellite-based detection of Canadian boreal forest fires: Development and application of the algorithm, *International Journal of Remote Sensing* **21**(16), 3057–3069.

[13] Moler, C. (2013) The Lambert W function, http://blogs.mathworks.com/cleve/2013/09/02/the-lambert-w-function/.

[14] Moubray, J. (1997) *Reliability-Centred Maintenance*, second edn, Butterworth-Heinemann, Oxford.

[15] NIST/SEMATECH (2018) *e-Handbook of Statistical Methods*, NIST.

[16] Takács, L. (1959) On a sojourn time problem in the theory of stochastic processes, *Transactions of the American Mathematical Society* **93**(3), 531–540.

[17] Takács, L. (1974) Sojourn time problems, *The Annals of Probability* **2**(3), 420–431.

[18] Trivedi, K. S. (2002) *Probability and Statistics with Reliability, Queuing, and Computer Science Applications*, second edn, John Wiley & Sons, New York.

[19] Wagner, D., Mylander, W. & Sanders, T. (1999) *Naval Operations Analysis*, 3rd edn, Naval Institute Press, Annapolis, Maryland.

[20] Yang, G., Dou, L., Chen, J. & Hou, C. (2001) Synergy decision in the multi-target tracking based on irst and intermittent-working radar, *Information Fusion* **2**(4), 243 – 250.

This page is intentionally blank

Appendix A. Proofs of Technical Results

Lemma 3. Let $\{F_\tau\}_\tau$ and $\{G_\tau\}_\tau$ be sequences of cumulative distribution functions, and $\{X_{F_\tau}\}_\tau$ and $\{X_{G_\tau}\}_\tau$ be the corresponding sequences of random variables. Suppose that for all τ, X_{F_τ} and X_{G_τ} are both continuous and non-negative, and $\mathbb{E}(X_{F_\tau})$ and $\mathbb{E}(X_{G_\tau})$ are both finite. Suppose further that for all $x \geq 0$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|F_\tau(x) - G_\tau(x)| < \epsilon$. Then for all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.

Proof of Lemma 3. In three steps:

1. *For any $\epsilon_F, \epsilon_G > 0$ there exist $a > 0$, $n > 0$ such that*

$$\left| \mathbb{E}(X_{F_\tau}) - \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) \right| < \epsilon_F$$
$$\left| \mathbb{E}(X_{G_\tau}) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right) \right| < \epsilon_G$$

   where $x_k = \frac{k}{n} \cdot a$.

   Proof. We have

$$\begin{aligned}
\mathbb{E}(X_{F_\tau}) &= \int_{[0,\infty)} x \, dF_(x) \, dx \\
&= \lim_{a\to\infty} \lim_{n\to\infty} \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) \\
\mathbb{E}(X_{G_\tau}) &= \int_{[0,\infty)} x \, dG_(x) \, dx \\
&= \lim_{a\to\infty} \lim_{n\to\infty} \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)
\end{aligned}$$

   Now $\mathbb{E}(X_{F_\tau})$, $\mathbb{E}(X_{G_\tau})$ are finite, so the limits exist and thus a, n exist by definition. □

2. *Let $a, n > 0$ and $x_k = \frac{k}{n} \cdot a$ for $k = 1 \ldots n$. For any $\epsilon_\Sigma > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then*

$$\left| \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right) \right| < \epsilon_\Sigma \qquad (1)$$

   Proof. For $k = 1 \ldots n$ put $\epsilon_k = \dfrac{\epsilon_\Sigma}{2n \cdot x_k}$. Then $\epsilon_k > 0$ so by assumption, for each k there exists $\tau'_{1,k}$ such that if $\tau > \tau'_{1,k}$ then $|F_\tau(x_k) - G_\tau(x_k)| < \epsilon_k$ and likewise there exists $\tau'_{2,k}$ such that if $\tau > \tau'_{2,k}$ then $|F_\tau(x_{k-1}) - G_\tau(x_{k-1})| < \epsilon_k$. Set $\tau' = \max\{\max\left(\tau'_{1,k}, \tau'_{2,k}\right) :$

$k = 1 \dots n\}$. Then for any $\tau > \tau'$

$$\begin{aligned}
&\left| \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right) \right| \\
&= \left| \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - G_\tau(x_k)\right) - \sum_{k=1}^{n} x_k \left(F_\tau(x_{k-1}) - G_\tau(x_{k-1})\right) \right| \\
&\le \sum_{k=1}^{n} x_k \left|F_\tau(x_k) - G_\tau(x_k)\right| + \sum_{k=1}^{n} x_k \left|F_\tau(x_{k-1}) - G_\tau(x_{k-1})\right| \\
&< \sum_{k=1}^{n} x_k \frac{\epsilon_\Sigma}{2n \cdot x_k} + \sum_{k=1}^{n} x_k \frac{\epsilon_\Sigma}{2n \cdot x_k} \\
&= \epsilon_\Sigma
\end{aligned}$$

□

3. *For all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.*

Proof. Choose $\epsilon_\Sigma, \epsilon_F, \epsilon_G > 0$ such that $\epsilon_\Sigma + \epsilon_F + \epsilon_G = \epsilon'$. Construct a, n as per step 1 and then construct τ' via step 2. Then for any $\tau > \tau'$

$$\begin{aligned}
|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| \le &\left| \mathbb{E}(X_{F_\tau}) - \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) \right| + \\
&\left| \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right) \right| + \\
&\left| \mathbb{E}(X_{G_\tau}) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right) \right| \\
< &\ \epsilon_F + \epsilon_\Sigma + \epsilon_G
\end{aligned}$$

□

Lemma 4. $\left|\mathbb{P}(U \le u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| = \left|\mathbb{P}(-\zeta U \le -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right|$.

Proof. Let $Y \sim \mathcal{N}(\mu_U, \sigma_U^2)$ then $-\zeta Y \sim \mathcal{N}(-\zeta\mu_U, \zeta^2\sigma_U^2)$ by properties of the normal distribution. Now $U \le u$, $Y \le u$ if and only if $-\zeta U \ge -\zeta u$, $-\zeta Y \ge -\zeta u$ so

$$\begin{aligned}
\left|\mathbb{P}(-\zeta U \le -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right| &= |\mathbb{P}(-\zeta U \le -\zeta u) - \mathbb{P}(-\zeta Y \le -\zeta u)| \\
&= |(1 - \mathbb{P}(-\zeta U \ge -\zeta u)) - (1 - \mathbb{P}(-\zeta Y \ge -\zeta u))| \\
&= |\mathbb{P}(U \le u) - \mathbb{P}(Y \le u)| \\
&= \left|\mathbb{P}(U \le u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right|
\end{aligned}$$

as required. □

Lemma 8. The function

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

attains its maximum value of

$$\breve{p}_n = e^{-n}\frac{n^n}{n!}$$

when $\lambda = n$, and is increasing if $\lambda < n$ and decreasing if $\lambda > n$.

Proof. We have

$$\begin{aligned} h_n'(\lambda) &= -e^{-\lambda}\frac{\lambda^n}{n!} + e^{-\lambda}\frac{n\lambda^{n-1}}{n!} \\ &= e^{-\lambda}\frac{\lambda^{n-1}}{n!}(n-\lambda) \end{aligned}$$

hence $h_n'(\lambda) < 0$ if $\lambda < n$, $h_n'(\lambda) = 0$ if $\lambda = n$, and $h_n'(\lambda) > 0$ if $\lambda > n$. Thus the point $\lambda = n$ is the global maximum. □

Lemma 9. Given

$$\kappa_n(p) = -\frac{(n!p)^{1/n}}{n}$$

the function

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

has pre-images

$$\begin{aligned} h_{0,n}^{-1}(p) &= -nW_0(\kappa_n(p)) \\ h_{-1,n}^{-1}(p) &= -nW_{-1}(\kappa_n(p)) \end{aligned}$$

respectively mapping from $[0, \breve{p}_n]$ to $[0, n]$ and from $[0, \breve{p}_n]$ to $[n, \infty)$, where W_0, W_{-1} are the Lambert W function on its 0, -1 branches.

Proof. Suppose

$$p = e^{-\lambda}\frac{\lambda^n}{n!}$$

then

$$\begin{aligned} n!p &= e^{-\lambda}\lambda^n \\ (n!p)^{\frac{1}{n}} &= e^{-\frac{\lambda}{n}}\lambda \\ \kappa_n(p) &= -\tfrac{\lambda}{n}e^{-\frac{\lambda}{n}} \\ -\tfrac{\lambda}{n} &= W(\kappa_n(p)) \\ \lambda &= nW(\kappa_n(p)) \end{aligned}$$

and result follows by applying the two branches of W. □

Lemma 10. The function

$$f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \quad \text{where } t, \lambda \geq 0$$

attains its maximum at $t = \dfrac{k-1}{\lambda}$.

Proof. We have

$$\begin{aligned} f'(t) &= \frac{\lambda^k}{(k-1)!} \left(t^{k-1}(-\lambda e^{-\lambda t}) + (k-1)t^{k-2}e^{-\lambda t} \right) \\ &= \frac{\lambda^k t^{k-2} e^{-\lambda t}}{(k-1)!} (k - 1 - \lambda t) \end{aligned}$$

hence $f'(t) < 0$ if $k - 1 < \lambda t$, $f'(t) = 0$ if $k - 1 = \lambda t$, and $f'(t) > 0$ if $k - 1 > \lambda t$. Thus the point $t = (k-1)/\lambda$ is the global maximum. □

DISTRIBUTION LIST

An Intermittent Sensor versus a Target that Emits Glimpses as a Homogenous Poisson Process

Patrick Chisan Hew

S&T Program

Chief of Joint and Operations Analysis Division	
Research Leader Maritime Capability Analysis - Dr Jim Smelt	1
Acting Group Leader Maritime Systems Analysis - Mr Jez Gray	1
Author(s): Dr Patrick Hew	1

DEFENCE SCIENCE AND TECHNOLOGY GROUP DOCUMENT CONTROL DATA			1. DLM/CAVEAT (OF DOCUMENT)
2. TITLE An Intermittent Sensor versus a Target that Emits Glimpses as a Homogenous Poisson Process		3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION) Document (U) Title (U) Abstract (U)	
4. AUTHORS Patrick Chisan Hew		5. CORPORATE AUTHOR Defence Science and Technology Group 506 Lorimer St, Fishermans Bend, Victoria 3207, Australia	
6a. DST GROUP NUMBER DST-Group–TN–1765	6b. AR NUMBER 017-158	6c. TYPE OF REPORT Technical Note	7. DOCUMENT DATE April 2018 Original release November 2019 Reissued with corrections
8. OBJECTIVE ID qAV22221	9. TASK NUMBER NAV 17/525	10. TASK SPONSOR Director General SEA1000	
11. MSTC Maritime Capability Analysis		12. STC Maritime Systems Analysis	
13. DOWNGRADING/DELIMITING INSTRUCTIONS `http://dspace.dsto.defence.gov.au/dspace/`		14. RELEASE AUTHORITY Chief, Joint and Operations Analysis Division	

15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT

Approved for Public Release

OVERSEAS ENQUIRIES OUTSIDE STATED LIMITATIONS SHOULD BE REFERRED THROUGH DOCUMENT EXCHANGE, PO BOX 1500, EDINBURGH, SA 5111

16. DELIBERATE ANNOUNCEMENT

No Limitations

17. CITATION IN OTHER DOCUMENTS

No Limitations

18. RESEARCH LIBRARY THESAURUS

probability theory, measures of effectiveness, sensors

19. ABSTRACT

This technical note considers a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time, the sensor has a probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in that interval. We devise closed-form approximations to the distributions for those probabilities, prove that the approximations become perfect as the time interval becomes infinitely long (asymptotic distributions, pointwise convergence), and compare the approximations with empirical results obtained from simulations. *Revised 2019*: Summary of changes at Executive Summary.