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An Intermittent Sensor versus a Target that Emits Glimpses as a Homogenous Poisson Process

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ABSTRACT

This technical note considers a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time, the sensor has a probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in that interval. We devise closed-form approximations to the distributions for those probabilities, prove that the approximations become perfect as the time interval becomes infinitely long (asymptotic distributions, pointwise convergence), and compare the approximations with empirical results obtained from simulations.

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Executive Summary

The work in this report was motivated by studies of operations to understand the performance that may be required from future systems; for example, unmanned aerial vehicles hunting for time-sensitive targets and submarines standing off from counter-detection. When collapsed to their essentials, the operations could be modelled in terms of a sensor that alternates between working and broken at random times, and is looking for a target that reluctantly gives away glimpses as a homogenous Poisson process. We refer to this situation as the *intermittent sensor homogenous glimpses model.*

In studies of such operations, the key measures of performance include the probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in the time interval. This report establishes that if we accept the intermittent sensor homogenous glimpses model then the measures of performance have approximations that are easy to calculate, and the approximations are close to reality when the time interval is long. So while the model is evidently an abstraction of real life, it can be sufficiently valid for a first, 'back of the envelope' analysis. Moreover the approximations provide insight into how performance will behave overall, something that can be difficult to obtain from stochastic simulation only.

The results can be applied to analysis of operations whenever the intermittent sensor homogenous glimpses model is a valid abstraction of the operation being studied. The analysis proves that the approximations become perfect in the technical sense of 'pointwise convergence' and uses simulation to compare the approximations with reality. The report will be of interest to analysts who are considering the intermittent sensor homogenous glimpses model for their work.

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Notation

	Detection of glimpses
$[0, \tau]$	Time interval
U	Cumulative duration that the sensor is working during $[0, \tau]$
μ_U	Mean of U
σ_U^2	Variance of U
z	Mean time between glimpses from the target
ζ	Mean rate at which the target gives away glimpses $= \frac{1}{z}$
$\hat{\zeta}$	Upper bound on ζ
	Sensor reliability
W_k	Time to failure on the kth cycle
μ_{F}	Mean of sensor's times to failure
σ_{F}^2	Variance in times to failure
B_k	Time to repair the failure in the kth cycle
μ_{R}	Mean of times to repair the sensor
σ_{R}^2	Variance in times to repair
c	Calculation parameter (Lemma 2)
w	Calculation parameter (Lemma 2)
	Measures of performance during an interval $[0, \tau]$
P_n	Probability of detecting n glimpses
P^k	Probability of detecting the k-th glimpse
P_n^k	Probability of detecting the k-th glimpse given n glimpses
	Conventional formalisms
$\mathbb{R}$	Real numbers
$\mathbb{P}(\cdot)$	Probability of $\cdot$
$\mathbb{E}(\cdot)$	Expected value of $\cdot$
$\mathcal{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
$\ln \mathcal{N}(\mu, \sigma)$	Log-normal distribution from $\mathcal{N}(\mu, \sigma^2)$
$H(\cdot; \mu, \sigma)$	Cumulative distribution function for $\mathcal{N}(\mu, \sigma^2)$
$h(\cdot; \mu, \sigma)$	Probability density function for $\mathcal{N}(\mu, \sigma^2)$
$W(\cdot)$	Lambert-W function
	Notation specific to Section 3
$h_n(\cdot)$	Probability mass function for Poisson distribution
$\breve{p}_n$	Maximum value attained by $h_n(\cdot)$
$\kappa_n(\cdot)$	Conversion function
	Notation specific to Section 4
$g(\cdot)$	Rate of reward

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1. Introduction

This report originated in studies of operations to understand the performance that may be required from future systems; for example, unmanned aerial vehicles hunting for time-sensitive targets [1] and submarines standing off from counter-detection [3]. When collapsed to their essentials, the operations could be modelled in terms of a sensor that alternates between working and broken at random times, and is looking for a target that gives away glimpses as a homogenous Poisson process. We refer to this situation as the *intermittent sensor homogenous glimpses model*.

In studies of such operations, the key measures of performance are based on the probability of detecting the target with the sensor over some interval of time. This report obtains approximations to those measures of performance that are easy to calculate, and proves that the approximations are close to reality when the time interval is long. So while the intermittent sensor homogenous glimpses model is evidently an abstraction of real life, it can be sufficiently valid for a first, 'back of the envelope' analysis. Moreover the approximations provide insight into how performance will behave overall, something that can be difficult to obtain from stochastic simulation only.

This report presents the intermittent sensor homogenous glimpses model and calculations arising from it as a reference for future work. It is aimed at analysts who are contemplating the model for their studies of operations. Section 2 works through the model and its validity. Section 3 considers the probability of detecting n glimpses during an interval of time, and in particular the case $n = 0$ (target not detected). Section 4 looks at the probability of detecting the k-th glimpse during an interval of time and of detecting the k-th glimpse when there are n during that interval. We conclude with advice on how analysts can apply the findings.

2. Intermittent Sensor Homogenous Glimpses

The intermittent sensor homogenous glimpses model consists of a sensor that is working or broken under an alternating renewal process versus a target that reluctantly gives away glimpses as a homogenous Poisson process (Figure 1). In detail:

- At any point in the time interval $[0, \tau]$, the sensor is either *working* or *broken*. It alternates between those two states in cycles where a *cycle* is a duration spent working followed by a duration spent broken.

- The target gives away glimpses as a homogenous Poisson process with mean time between glimpses z. If a glimpse arrives when the sensor is working then the glimpse will be detected, but if the sensor is broken then the glimpse will be missed. We put $\zeta = \frac{1}{z}$ as the *mean glimpse rate* (mean rate at which the target gives away glimpses).

- Let W_k denote the duration that the sensor is working on the kth cycle. The durations $W_1, W_2, \ldots$ are positive, and are identically distributed with mean μ_F and variance σ_F^2 where μ_F and σ_F are both finite and positive.

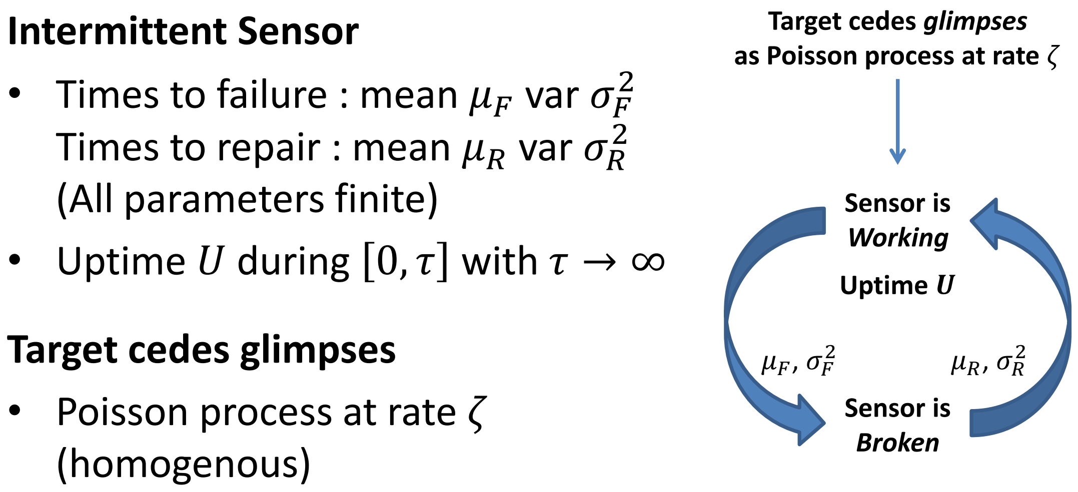

Figure 1: The intermittent sensor homogenous glimpses model.

- Likewise let B_k denote the duration that the sensor is broken on the kth cycle. The durations $B_1, B_2, \ldots$ are positive, and are identically distributed with mean μ_R and variance σ_R^2 where μ_R and σ_R are both finite and positive.

- The cycle durations $\{W_k + B_k : k = 1, 2, \ldots\}$ are mutually independent. To be clear, B_k is allowed to depend on W_k.

In any given application, it will be necessary to check that the model is valid abstraction of reality as opposed to being a 'strawman'. Working through the assumptions:

- That the sensor is either working or broken is a gross simplification of the real world. An accurate model would vary z with the distance of the sensor to the target, the target's susceptibility to detection (its signature), the environment, and other factors. The assumption of z constant (a homogenous Poisson process) can nonetheless provide a first-order insight into operations. The practical interest is in z large, namely a target that is difficult to detect as it rarely gives away glimpses. Modelling targets as giving away glimpses as a Poisson process (possibly non-homogenous) follows common practice in studies of search and screening (see [4, 8] for example).

- In assuming that the cycle durations are mutually independent, we implicitly assume that the sensor 'resets' with each cycle. The assumption is reasonable in the absence of opposing arguments.

It is well-known [7] that an alternating renewal process will forget the state that it was in at time $t = 0$, in that as time passes the probability of being in the working state approaches the *stationary probability* $\frac{\mu_\mathrm{F}}{\mu_\mathrm{R}+\mu_\mathrm{F}}$. Moreover the process is *ordinary* if it is working at $t = 0$ versus *in equilibrium* if its probability of being in the working state at $t = 0$ is the stationary probability. The model itself makes no assumptions about the sensor's state at $t = 0$. For adherence to real-world conditions it is arguably more realistic to assume that the sensor is working at the start of an operation. That said, one might assume that the sensor is allowed to 'run in' a bit and hence the process can be taken as being in equilibrium.

As notation, let $\mathbb{R}$ denote the real numbers, $\mathbb{P}(\cdot)$ be 'the probability of $\cdot$' and $\mathbb{E}(\cdot)$ be 'the expected value of $\cdot$'. Write $\mathcal{N}(\mu, \sigma^2)$ for the normal distribution with mean μ and variance σ^2. We will also use the following lemma; it establishes that if we can approximate the distribution of a random variable then we can use that approximation to estimate the variable's mean.

Lemma 1. Let $\{F_\tau\}_\tau$ and $\{G_\tau\}_\tau$ be sequences of cumulative distribution functions, and $\{X_{F_\tau}\}_\tau$ and $\{X_{G_\tau}\}_\tau$ be the corresponding sequences of random variables. Suppose that for all τ, X_{F_τ} and X_{G_τ} are both continuous and non-negative, and $\mathbb{E}(X_{F_\tau})$ and $\mathbb{E}(X_{G_\tau})$ are both finite. Suppose further that for all $x \geq 0$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|F_\tau(x) - G_\tau(x)| < \epsilon$. Then for all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.

Proof. See Appendix. □

3. Detecting n Glimpses

This section considers the probability P_n of detecting n glimpses during the time interval $[0, \tau]$. As notation for this section, let $\ln\mathcal{N}(\mu, \sigma)$ denote the log-normal distribution associated with $\mathcal{N}(\mu, \sigma^2)$. If $Y \sim \mathcal{N}(\mu, \sigma^2)$ then we write $\mathcal{N}(y; \mu, \sigma) = \mathbb{P}(Y \leq y)$. Similarly if $X \sim \ln\mathcal{N}(\mu, \sigma)$ then $\ln\mathcal{N}(y; \mu, \sigma) = \mathbb{P}(Y \leq y)$. Our analysis hinges on the following lemma.

Lemma 2 (Uptime is asymptotically normal)**.** Let U be the *uptime*, namely the cumulative duration that the sensor is working during an interval $[0, \tau]$. Then as $\tau \to \infty$, the quantity $\dfrac{U - \mu_U}{\sigma_U}$ converges in distribution to $\mathcal{N}(0, 1)$ where

$$\begin{aligned}
\mu_U &= c\tau \\
\sigma_U^2 &= 2c(1-c)w\tau \\
c &= \frac{\mu_\mathrm{F}}{\mu_\mathrm{R} + \mu_\mathrm{F}} \\
w &= \frac{\mu_\mathrm{R}^2\sigma_\mathrm{F}^2 + \mu_\mathrm{F}^2\sigma_\mathrm{R}^2}{2\mu_\mathrm{R}\mu_\mathrm{F}(\mu_\mathrm{R} + \mu_\mathrm{F})}
\end{aligned}$$

Proof. As μ_F, μ_R, σ_F, σ_R are all finite, we may apply the classic finding in reliability theory [6]. Result then follows from algebraic manipulations that assume $\mu_\mathrm{F}, \mu_\mathrm{R} > 0$. □

In short, U is asymptotically normal. Meanwhile, the target is giving away glimpses as a homogenous Poisson process at rate ζ. Hence the number of glimpses during the time interval will be a random variable that we can calculate as ζU *and* that random variable will also be asymptotically normal. We formalize this idea in the following two lemmas as the stepping stone to the main results for this section.

Lemma 3. $\left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| = \left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right|$.

Proof. See Appendix. □

Lemma 4 (Number of glimpses is asymptotically normal)**.** For any $u \in \mathbb{R}$, $\epsilon > 0$:

1. There exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(\zeta U \leq \zeta u) - \mathcal{N}(\zeta u; \zeta \mu_U, \zeta^2 \sigma_U^2)\right|$.

2. There exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta \mu_U, \zeta^2 \sigma_U^2)\right|$.

Proof. Take Lemma 2 and unpack the definition of convergence in distribution: for any $u \in \mathbb{R}$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then

$$\left|\mathbb{P}\left(\frac{U - \mu_U}{\sigma_U} \leq \frac{u - \mu_U}{\sigma_U}\right) - \mathcal{N}(u; 0, 1)\right| < \epsilon$$

By properties of the normal distribution this holds if and only if

$$\left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| < \epsilon$$

Hence for result (1) we apply properties of the normal distribution to get

$$\left|\mathbb{P}(\zeta U \leq \zeta u) - \mathcal{N}(\zeta u; \zeta \mu_U, \zeta^2 \sigma_U^2)\right| < \epsilon$$

Meanwhile for result (2) we apply Lemma 3 to get

$$\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta \mu_U, \zeta^2 \sigma_U^2)\right| < \epsilon$$

□

3.1. Detecting Zero Glimpses

The following result deduces that the distribution of P_0 approaches log-normal as $\tau \to \infty$.

Proposition 1. Set $\mu = -\zeta \mu_U$, $\sigma^2 = \zeta^2 \sigma_U^2$. For any $0 \leq p \leq 1$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $\left|\mathbb{P}(P_0 \leq p) - \ln \mathcal{N}(p; \mu, \sigma^2)\right| < \epsilon$.

Proof. Suppose $\epsilon > 0$. Given p, construct $u = -\frac{1}{\zeta}\ln(p)$ so $p = e^{-\zeta u}$. By Lemma 4 there exists $\tau' > 0$ such that if $\tau > \tau'$ then

$$\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; \mu, \sigma^2)\right| < \epsilon$$

Now $P_0 = \mathbb{P}(\text{Zero glimpses acquired during } [0, \tau]) = e^{-\zeta U}$. Moreover the function $\exp(\cdot)$ is strictly increasing so $-\zeta U \leq -\zeta u$ if and only if $P_0 \leq p$. Hence $\mathbb{P}(P_0 \leq p) = \mathbb{P}(-\zeta U \leq -\zeta u)$ so if $\tau > \tau'$ then

$$\left|\mathbb{P}(P_0 \leq p) - \mathcal{N}(\ln p; \mu, \sigma^2)\right| < \epsilon$$

or equivalently

$$|\mathbb{P}(P_0 \leq p) - \ln \mathcal{N}(p; \mu, \sigma)| < \epsilon$$

□

We immediately obtain an approximation to the expected value of P_0 that becomes perfect as $\tau \to \infty$.

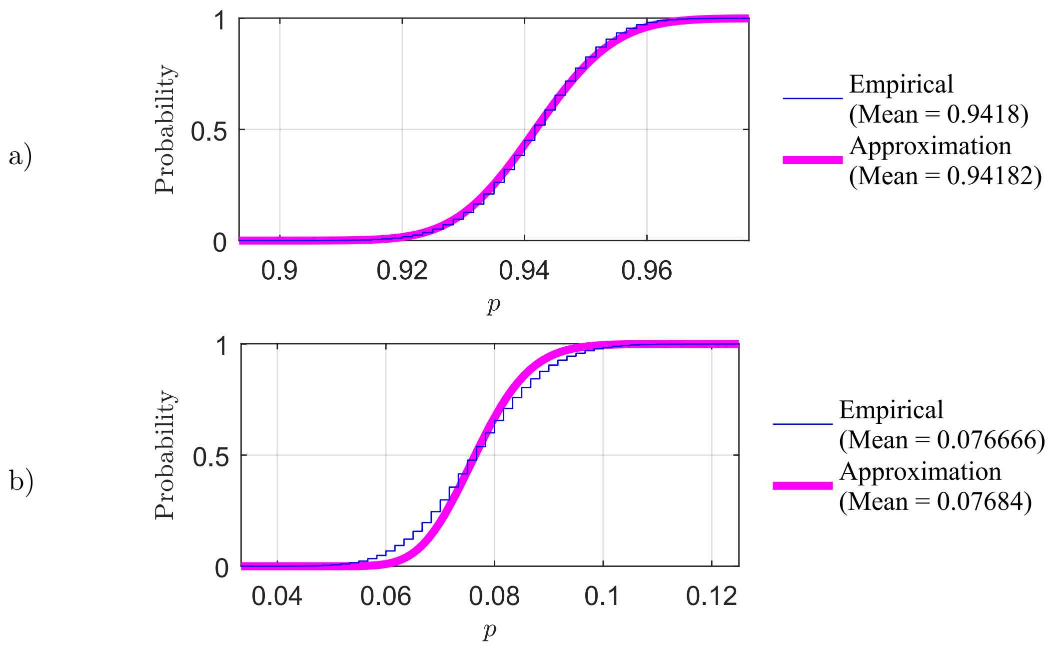

Figure 2: Approximation to distribution of P_0 vs empirical distribution from simulations. Times to fail follow $\mathrm{Exp}(0.3)$*, times to repair follow* $\mathrm{Exp}(0.7)$*, a)* $z = 150, \tau = 30$*. b)* $z = 70, \tau = 600$*.*

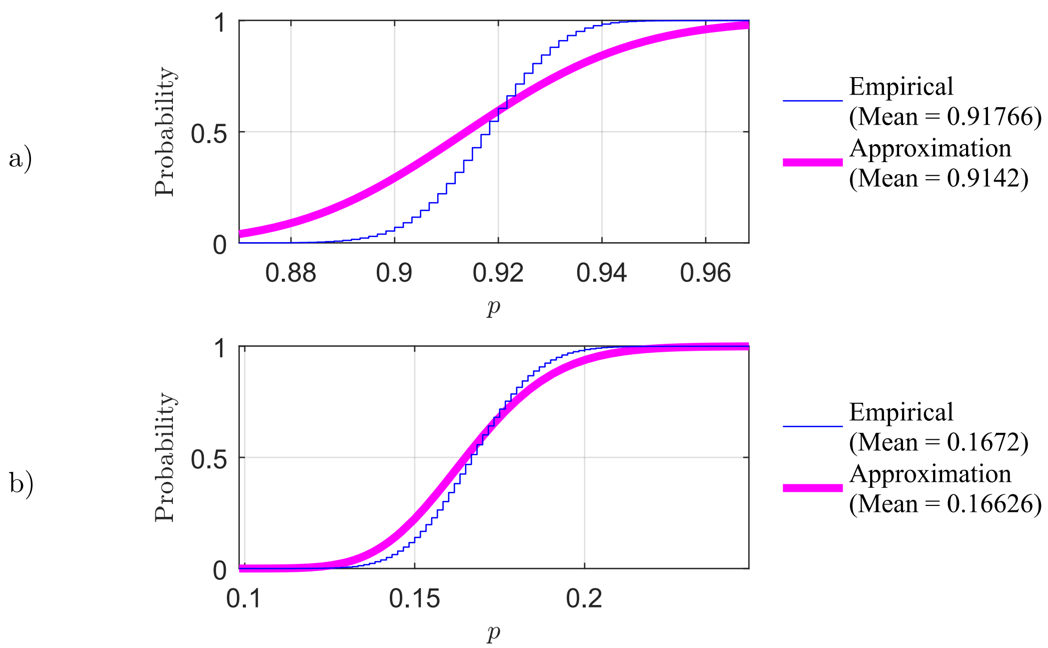

Figure 3: Approximation to distribution of P_0 vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*, a)* $z = 150, \tau = 30$*. b)* $z = 70, \tau = 600$*.*

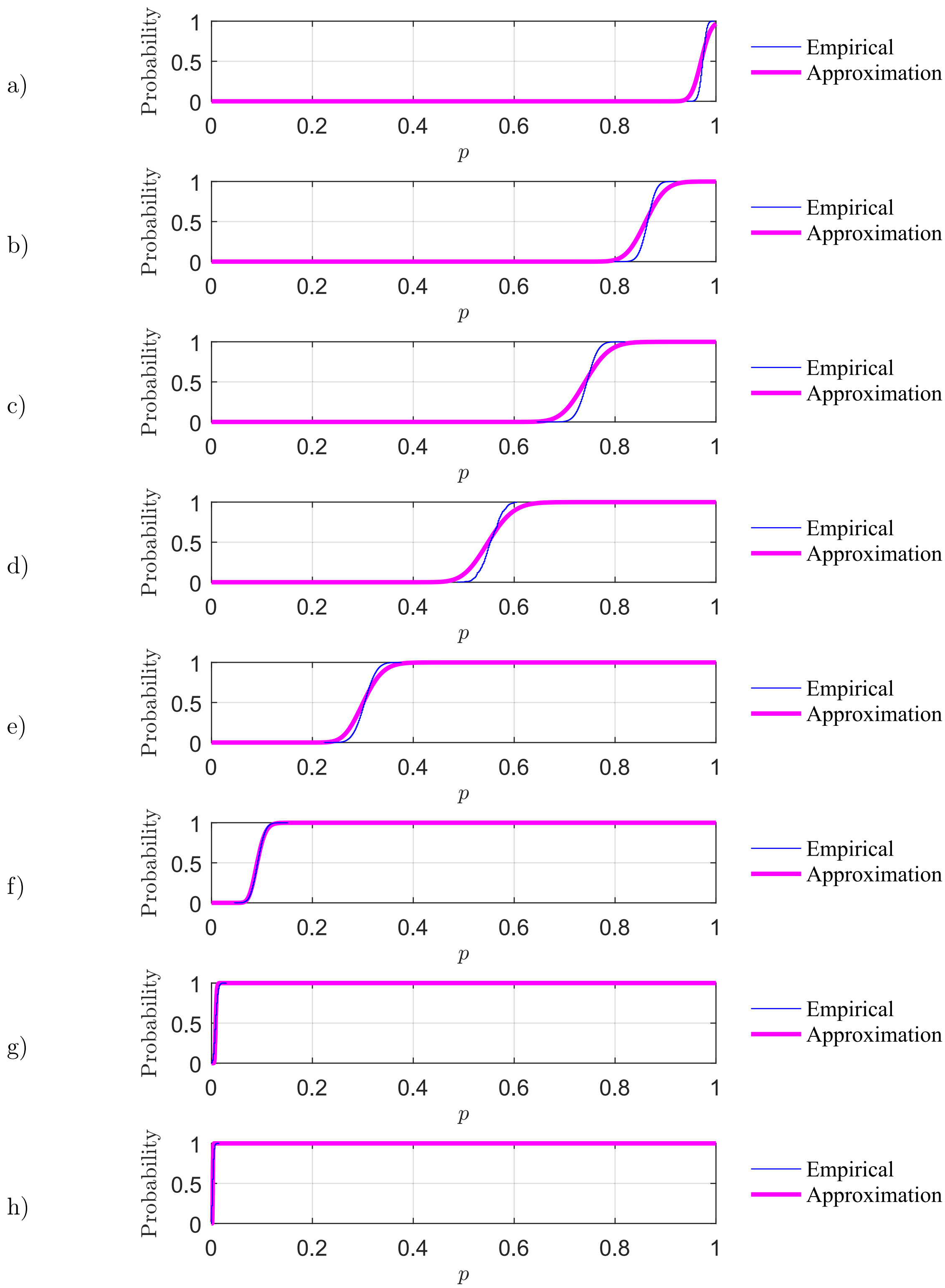

Figure 4: Approximation to distribution of P_0 vs empirical distribution from simulations. Times to fail follow $\ln\mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, a) $\tau = 10$. b) $\tau = 50$. c) $\tau = 100$. d) $\tau = 200$. e) $\tau = 400$. f) $\tau = 800$. g) $\tau = 1600$. h) $\tau = 2000$.

Corollary 1. Put

$$\nu_\tau = \exp\left(\zeta c\tau(\zeta(1-c)w - 1)\right)$$

Then $|\mathbb{E}(P_0) - \nu_\tau| \to 0$ as $\tau \to \infty$ provided $\zeta \le \hat{\zeta}$ where

$$\hat{\zeta} = \frac{1}{2(1-c)w}$$

Proof. Let $X \sim \ln \mathcal{N}(\mu, \sigma)$. Lemma 1 finds that $|\mathbb{E}(P_0) - \mathbb{E}(X)| \to 0$ as $\tau \to \infty$. We then use known properties of the log-normal distribution to calculate

$$\begin{aligned}\mathbb{E}(X) = e^{\mu+\sigma^2/2} &= \exp\left(-\zeta c\tau + \zeta^2 c(1-c)w\tau\right)\\ &= \exp\left(\zeta c\tau(\zeta(1-c)w - 1)\right)\end{aligned}$$

We now describe why the caveat $\zeta \le \hat{\zeta}$ is required. Observe that $\exp(\cdot)$ is an increasing function and $f(\zeta) = \zeta c\tau(\zeta(1-c)w-1)$ is a positive quadratic with inflection point $\hat{\zeta}$. Hence f is decreasing on $\zeta \le \hat{\zeta}$, but increasing thereafter. But in reality, we should have $\mathbb{E}(X) \to 0$ as $\zeta \to \infty$: if the target is giving away a huge number of glimpses then it is bound to be acquired by the sensor, whereby $P_0 \to 0$ surely and hence $X \to 0$ surely. Thus we use the caveat to constrain ζ to the domain on which f is decreasing. □

Remark. The approximation to $\mathbb{E}(P_0)$ provided by Corollary 1 is an easily-calculated measure of performance for the sensor. Indeed $1 - P_0$ is the probability of detecting the target.

The underlying issue that leads to the caveat is that we have taken limits in a non-commutative order. The quantity ζu corresponds physically to the number of glimpses given away by the target during u. As $\zeta \to \infty$, we should see $\zeta u \to \infty$ surely, but instead we are tied to a normal distribution with mean $\mu \to -\infty$ and deviation $\sigma \to \infty$. The correct treatment takes $\zeta \to \infty$ first, and then $\tau \to \infty$. The caveat is for mathematical correctness. The practical interest is in ζ small, wherein the target gives away glimpses rarely.

Figures 2 through 4 compare Proposition 1's approximation to the distribution of P_0 with empirical results from simulation. Each simulation run represented the sensor working intermittently during some interval $[0, \tau]$. The sensor had probability $\frac{\mu_F}{\mu_R+\mu_F}$ of being working at time $t = 0$ (simulation of the equilibrium process). A total of 4,000,000 runs were generated, and then P_0 was estimated 50,000 times. Each estimate used 600 of the runs, in sampling without replacement. Each figure shows the empirical cumulative distribution function from the 50,000 estimations, compared with the proposed approximation.

The simulations show agreement between the approximation and empirical values for $\mathbb{E}(P_0)$ (Corollary 1). The distributions for P_0 match our intuitions: when $\tau = 0$ we have $P_0 = 0$ and as $\tau \to \infty$ we have $P_0 \to 1$ surely. While there is discrepancy between the approximate and empirical distributions, it disappears as $\tau \to \infty$.

3.2. Detecting One or More Glimpses

The following result establishes that when $n \geq 1$ the distribution of P_n can be thought of as approaching 'Lambert W-normal' as $\tau \to \infty$. For any positive integer n and $0 \leq p \leq 1$ put

$$\kappa_n(p) = -\frac{(n!p)^{1/n}}{n}$$

Let h be the probability mass function for the Poisson distribution, namely

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

It is readily shown that $h_n(\lambda)$ attains its maximum value of

$$\breve{p}_n = e^{-n}\frac{n^n}{n!}$$

when $\lambda = n$, and is increasing if $\lambda < n$ and decreasing if $\lambda > n$ (Appendix, Lemma 7). The function has two pre-images, namely

$$h_{0,n}^{-1}(p) = -nW_0(\kappa_n(p))$$
$$h_{-1,n}^{-1}(p) = -nW_{-1}(\kappa_n(p))$$

respectively mapping from $[0, \breve{p}_n]$ to $[0, n]$ and from $[0, \breve{p}_n]$ to $[n, \infty)$, where W_0, W_{-1} are the Lambert W function on its 0, -1 branches (Appendix, Lemma 8).

Proposition 2. Set $\mu = -\zeta\mu_U$, $\sigma^2 = \zeta^2\sigma_U^2$ and

$$F_\tau(p) = 1 - \mathcal{N}\big(W_0(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big) + \mathcal{N}\big(W_{-1}(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big)$$

For any $0 \leq p \leq 1$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{P}(P_n \leq p) - F_\tau(p)| < \epsilon$.

Proof. Suppose $\epsilon > 0$. Given p, construct $u_0 = \frac{1}{\zeta}h_{0,n}^{-1}(p)$, $u_{-1} = \frac{1}{\zeta}h_{-1,n}^{-1}(p)$ so $p = h_n(\zeta u_0)$ and $p = h_n(\zeta u_{-1})$. By Lemma 4 there exists $\tau_0' > 0$ such that if $\tau > \tau_0'$ then

$$\big|\mathbb{P}(\zeta U \leq \zeta u_0) - \mathcal{N}(\zeta u_0; \zeta\mu_U, \zeta^2\sigma_U^2)\big| < \tfrac{1}{2}\epsilon$$

and likewise there exists $\tau_{-1}' > 0$ such that if $\tau > \tau_{-1}'$ then

$$\big|\mathbb{P}(\zeta U \leq \zeta u_{-1}) - \mathcal{N}(\zeta u_{-1}; \zeta\mu_U, \zeta^2\sigma_U^2)\big| < \tfrac{1}{2}\epsilon$$

Put $\tau' = \max(\tau_0', \tau_{-1}')$. Now $P_n = \mathbb{P}(n \text{ glimpses acquired during } [0, \tau]) = h_n(\zeta U)$. Moreover $h_n(\cdot)$ is increasing on $[0, \breve{p}_n]$ and decreasing on $[\breve{p}_n, \infty)$ so $P_n \leq p$ if and only if $\zeta U \leq \zeta u_0$ or $\zeta U \geq \zeta u_{-1}$. Hence $\mathbb{P}(P_n \leq p) = \mathbb{P}(\zeta U \leq \zeta u_0) + \mathbb{P}(\zeta U \geq \zeta u_{-1})$ so if $\tau > \tau'$ then

$$\Big|\mathbb{P}(P_n \leq p) - (\mathcal{N}(h_{0,n}^{-1}(p); \zeta\mu_U, \zeta^2\sigma_U^2) + 1 - \mathcal{N}(h_{-1,n}^{-1}(p); \zeta\mu_U, \zeta^2\sigma_U^2))\Big| < \epsilon$$

or equivalently

$$\big|\mathbb{P}(P_n \leq p) - \big(1 - \mathcal{N}\big(W_0(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big) + \mathcal{N}\big(W_{-1}(\kappa_n(p)); \tfrac{1}{n}\mu, \tfrac{1}{n^2}\sigma^2\big)\big)\big| < \epsilon$$

□

As before, we get an approximation to the expected value of $\breve{p}_n$ that becomes perfect as $\tau \to \infty$.

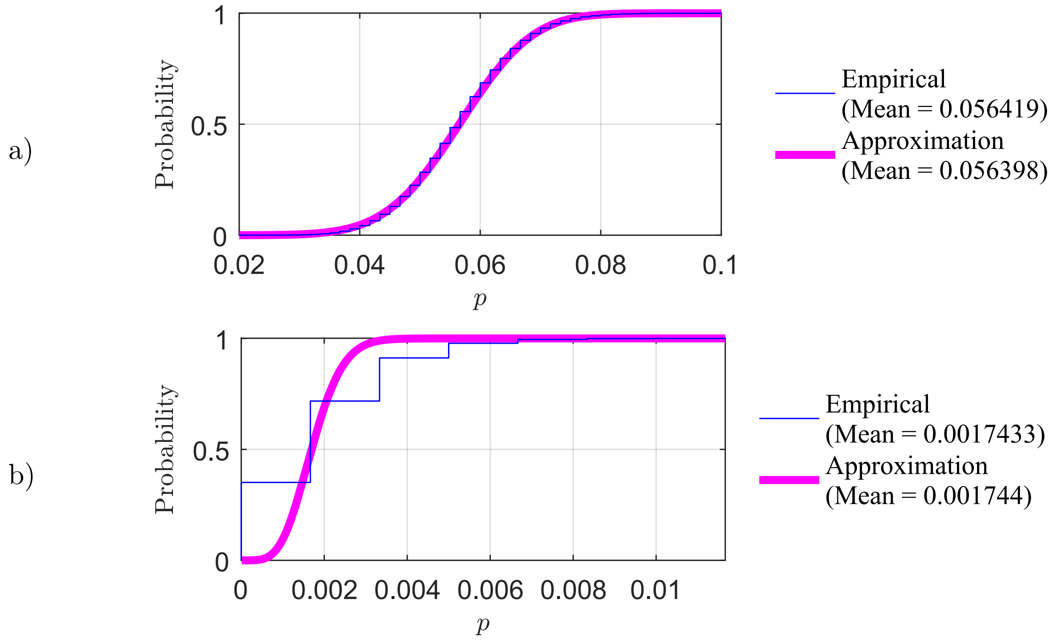

Figure 5: Approximation to distribution of P_n *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 30$ *a)* $n = 1$*. b)* $n = 2$*.*

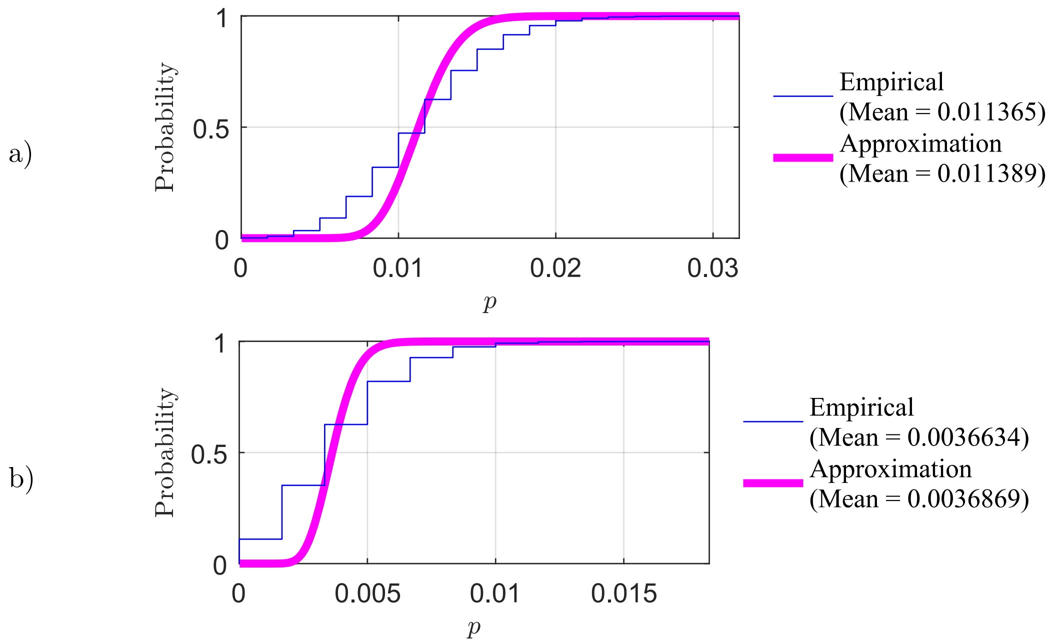

Figure 6: Approximation to distribution of P_n *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 600$ *a)* $n = 7$*. b)* $n = 16$*.*

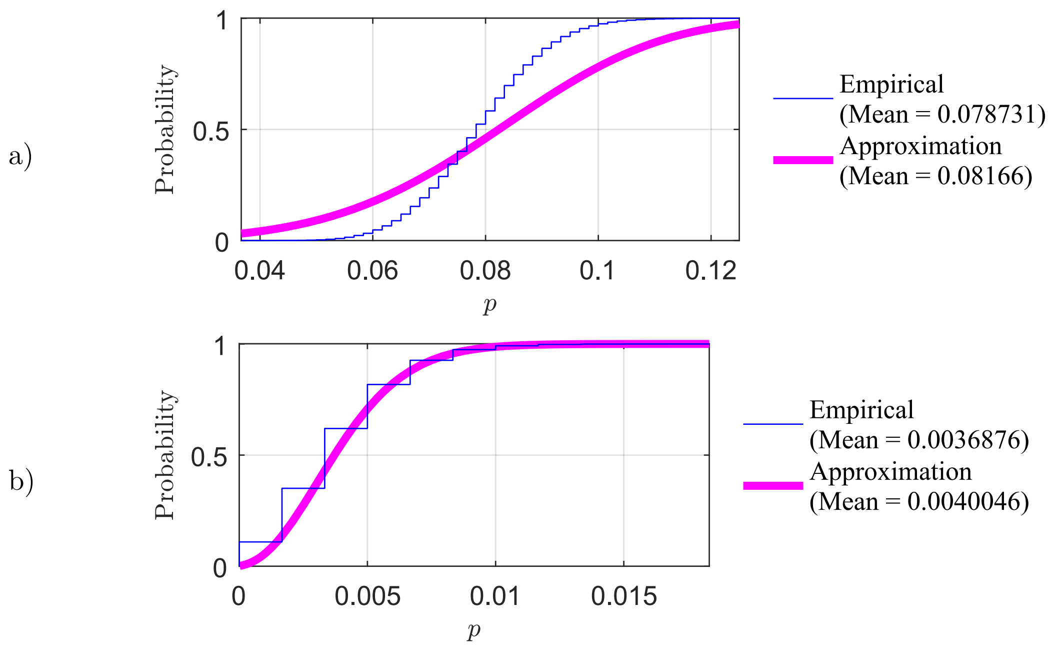

Figure 7: Approximation to distribution of P_n vs empirical distribution from simulations. Times to fail follow $\ln\mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, $\tau = 30$ a) $n = 1$. b) $n = 2$.

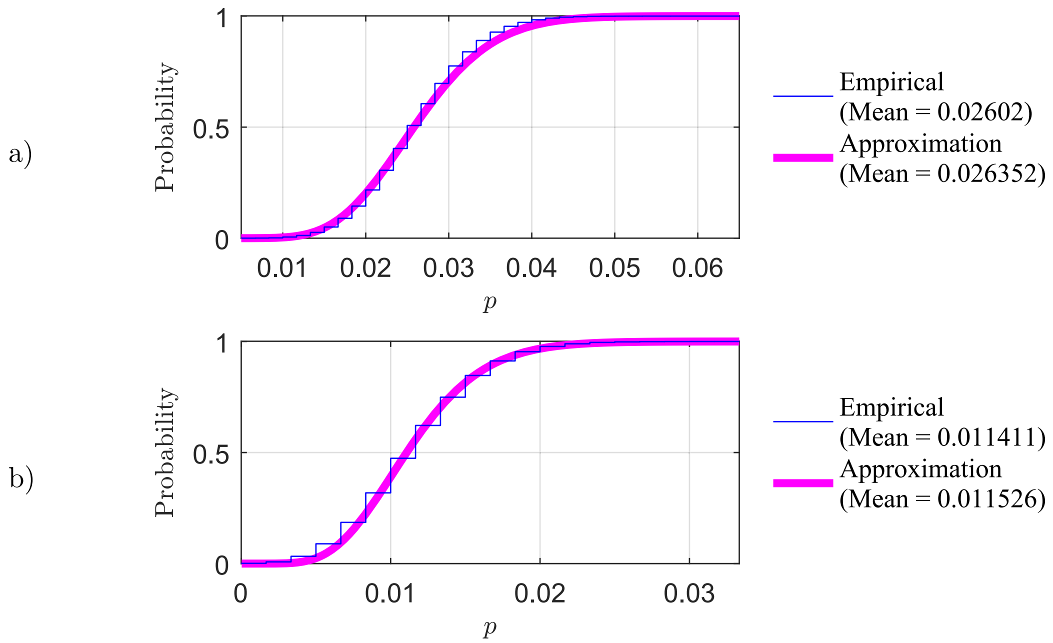

Figure 8: Approximation to distribution of P_n vs empirical distribution from simulations. Times to fail follow $\ln\mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, $\tau = 600$ a) $n = 8$. b) $n = 9$.

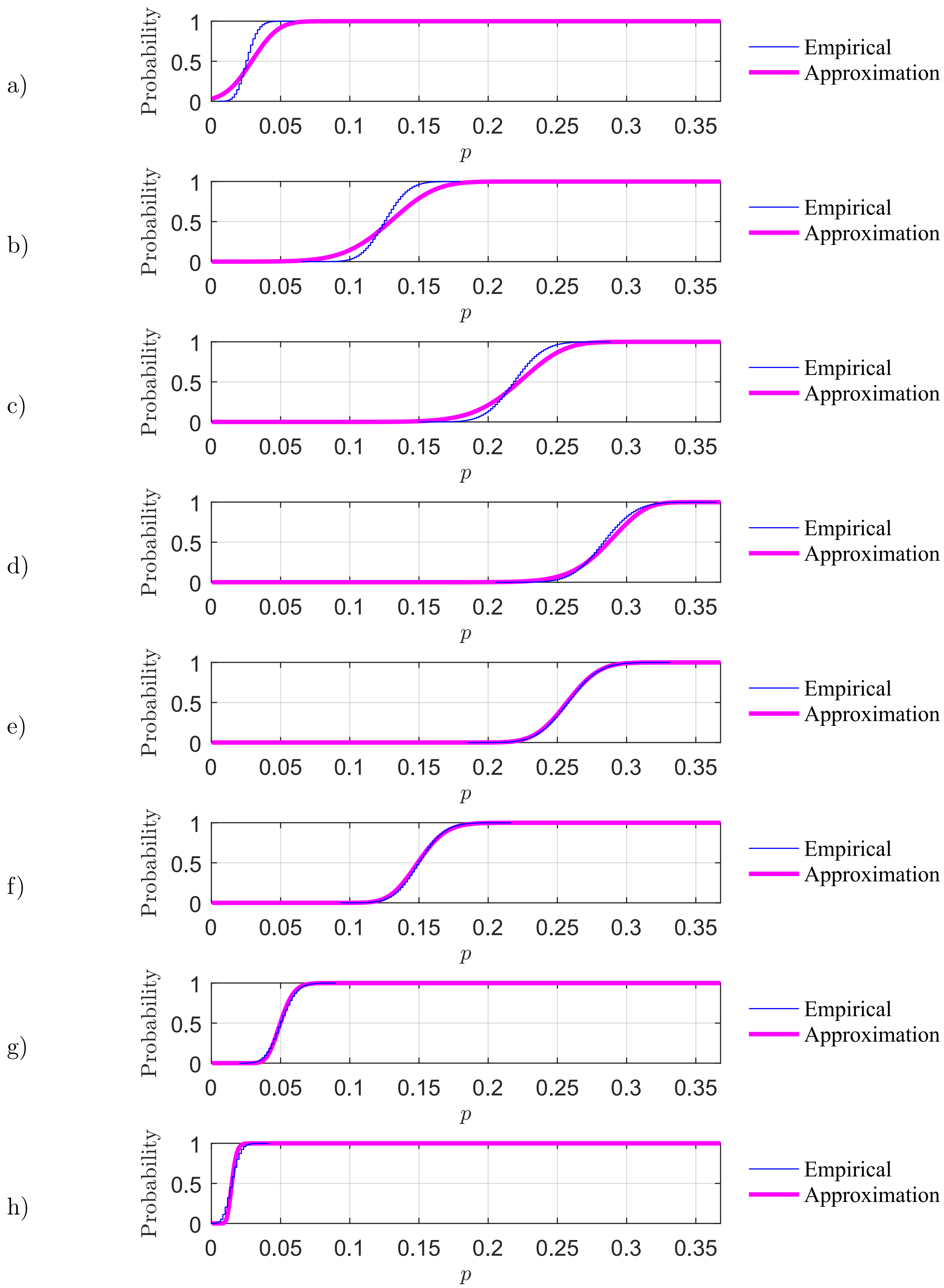

Figure 9: Approximation to distribution of P_1 vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, a) $\tau = 10$. b) $\tau = 50$. c) $\tau = 100$. d) $\tau = 150$. e) $\tau = 700$. f) $\tau = 1000$. g) $\tau = 1500$. h) $\tau = 2000$.

Corollary 2. Put

$$\nu_\tau = \breve{p}_n - \int_0^{\breve{p}_n} F_\tau(p)\,dp$$

Then $|\mathbb{E}(P_n) - \nu_\tau| \to 0$ as $\tau \to \infty$.

Proof. Let $X \sim F_\tau$. Lemma 1 finds that $|\mathbb{E}(P_n) - \mathbb{E}(X)| \to 0$ as $\tau \to \infty$. Now it is well-known that $\mathbb{E}(X) = \int_0^\infty (1 - F_\tau(p))\,dp$ and the domain of F_τ is $[0, \breve{p}_n]$. □

Figures 5 through 9 compare Proposition 2's approximation to the distribution of P_n with empirical results from simulation. The simulations were conducted as for the $n = 0$ case. The approximate (Corollary 1) and empirical values for $\mathbb{E}(P_n)$ are close. The distributions for P_n match our intuitions: when $\tau = 0$ we have $P_n = 0$. As τ increases, P_n initially concentrates around the value $\breve{p}_n$, but then as $\tau \to \infty$ we have $P_n \to 1$ surely. The discrepancy between the approximate and empirical distributions disappears as $\tau \to \infty$.

It is worth noting that when P_n is concentrated around $\breve{p}_n$, the proposed approximation to the distribution of P_n can take a long time to evaluate. The reason is that if $p \approx \breve{p}_n$ then $\kappa_n(p) \approx -\frac{1}{e}$ but naive implementations of W can take a long time to converge to accurate answers in this neighbourhood [5]. The issue can be addressed by using a careful implementation of W.

4. Detecting the k-th Glimpse

This section considers the probability P^k of detecting the k-th glimpse during the time interval $[0, \tau]$ and the probability P_n^k of detecting the k-th glimpse given that the target gives away n glimpses during that time interval. Our analysis hinges on the following lemma.

Lemma 5 (Accumulated reward is asymptotically normal)**.** Let X_t denote the sensor's state at time t wherein $X_t = 0$ if the sensor is broken and $X_t = 1$ if it is working. Given $g : [0,1] \to \mathbb{R}$, put $Q = \int_0^\tau g(t/\tau) X_t\,dt$ (reward the sensor at rate $g(t/\tau)$ if it is working at time t), and set

$$\mu_Q = \bar{g}\mu_U$$
$$\sigma_Q^2 = \gamma\sigma_U^2$$

where $\bar{g} = \int_0^1 g(x)\,dx$, $\gamma = \int_0^1 (g(x))^2\,dx$, and μ_U, σ_U^2 are provided by Lemma 2. Suppose that all of the following conditions are satisfied:

- $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty, \mathbb{E}(B_k^3) < \infty$, for all k.
- $-\infty < \bar{g} < \infty$, $0 < \gamma < \infty$, and $|\int_0^1 g(x)g'(x)\,dx| < \infty$.

Then as $\tau \to \infty$, the quantity $\dfrac{Q - \mu_Q}{\sigma_Q}$ converges in distribution to $\mathcal{N}(0,1)$.

Proof. See [2]. □

We will use the following corollary to find the probabilities of interest.

Corollary 3. Suppose that the waiting time to a glimpse has probability density function f. If $g(s) = f(s\tau)$ then Q approximates the probability of seeing that glimpse.

Proof. During any infinitesimal interval $[t, t + \delta t]$, the glimpse will be detected at probability $f(t)$ if the sensor is working at time t. Hence the probability of seeing the glimpse is the accumulation of those probabilities over the full interval $[0, \tau]$. Accordingly, we construct g so that $s\tau$ progresses across $[0, \tau]$ as s progresses across $[0, 1]$. □

Remark. Corollary 3 applies to any arrival process, not just the homogenous Poisson one.

4.1. Detecting the k-th Glimpse

We can immediately deduce an approximation to P^k.

Proposition 3. To obtain an approximation to P^k, apply Lemma 5 with $g(s) = f(s\tau; k, \zeta)$ where f is the probability density function for the Erlang distribution with rate parameter λ

$$f(t; k, \lambda) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \quad \text{where } t, \lambda \geq 0$$

Proof. The Erlang distribution provides the waiting time to the k-th arrival in a homogeneous Poisson process on $[0, \infty)$. Result follows from Corollary 3. □

Figures 10 through 14 compare Proposition 3's approximation to the distribution of P^k with empirical results from simulation. The simulations were conducted as for the previous section. The approximate and empirical values for $\mathbb{E}(P^k)$ are close. The discrepancy between the approximate and empirical distributions disappears as $\tau \to \infty$. Intuitively, and as seen in the empirical distribution, we need τ large enough to have a chance of seeing the k-th glimpse; indeed $f(t; k, \zeta)$ attains its maximum at $t = (k-1)/\zeta = (k-1)z$ (Appendix, Lemma 9). Increasing τ beyond this value will not affect the probability – for a greater chance of seeing the glimpse, the sensor needs to be working more during $[0, \tau]$.

4.2. Detecting the k-th Glimpse Given n Glimpses

We now deduce an approximation to P_n^k. The result follows from the following lemma.

Lemma 6. The function

$$f_n(t; k, \zeta, \tau) = \frac{n!}{(k-1)!\,(n-k)!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^{n-k}$$

is the probability density function for the waiting time to the k-th arrival in a Poisson process on $[0, \infty)$ with rate parameter ζ given n glimpses during the time interval $[0, \tau]$.

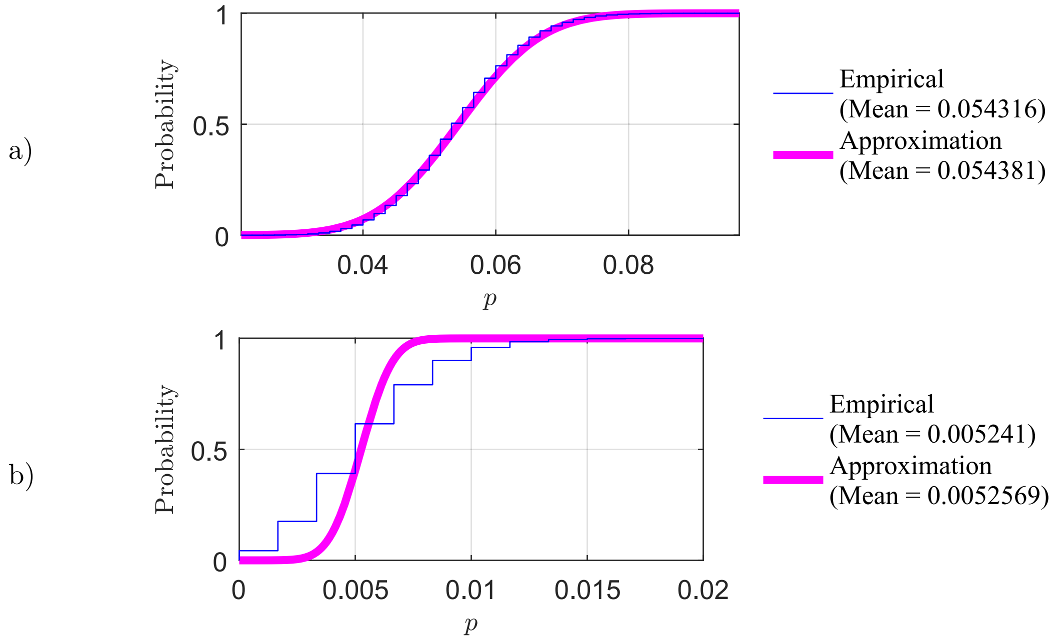

Figure 10: Approximation to distribution of P^k *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 30$ *a)* $k = 1$*. b)* $k = 2$*.*

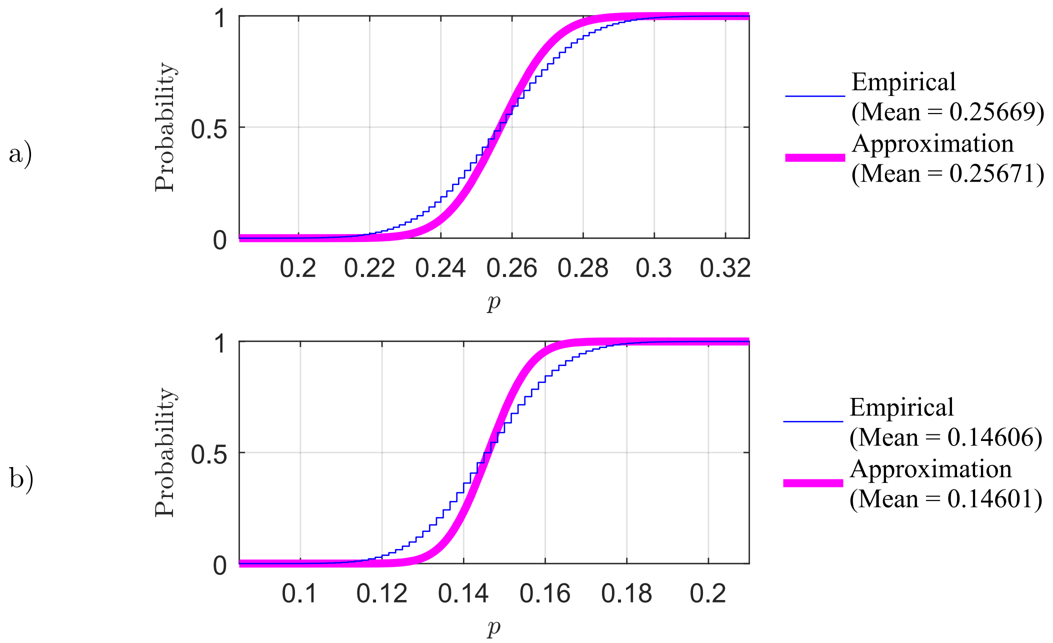

Figure 11: Approximation to distribution of P^k *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 600$ *a)* $k = 6$*. b)* $k = 9$*.*

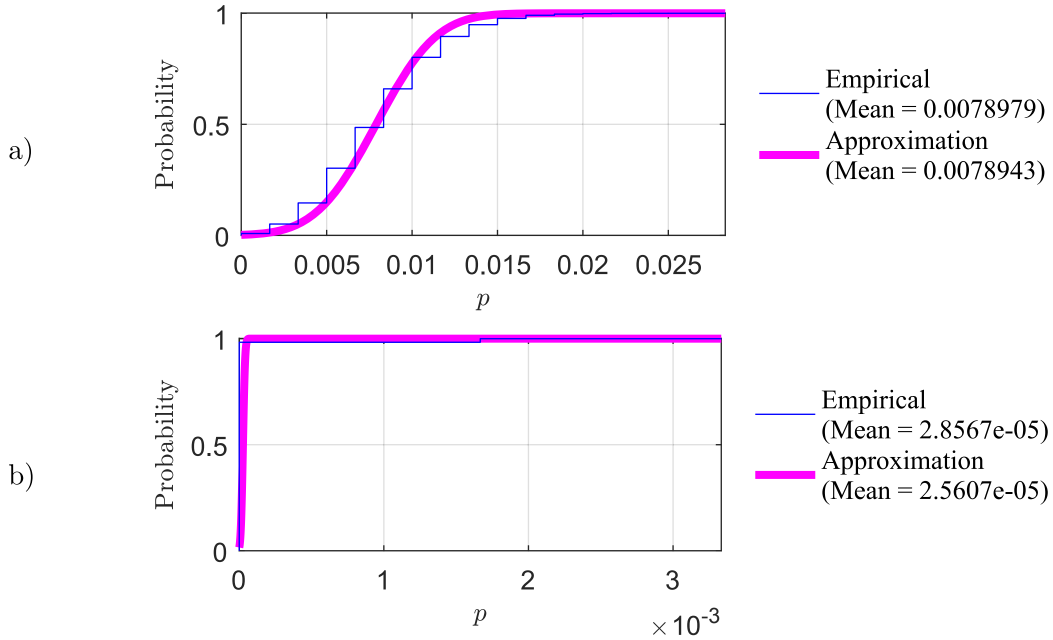

Figure 12: Approximation to distribution of P^k vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$, *times to repair are uniformly distributed on* $[2, 4]$, $z = 150$, $\tau = 30$ *a)* $k = 2$. *b)* $k = 4$.

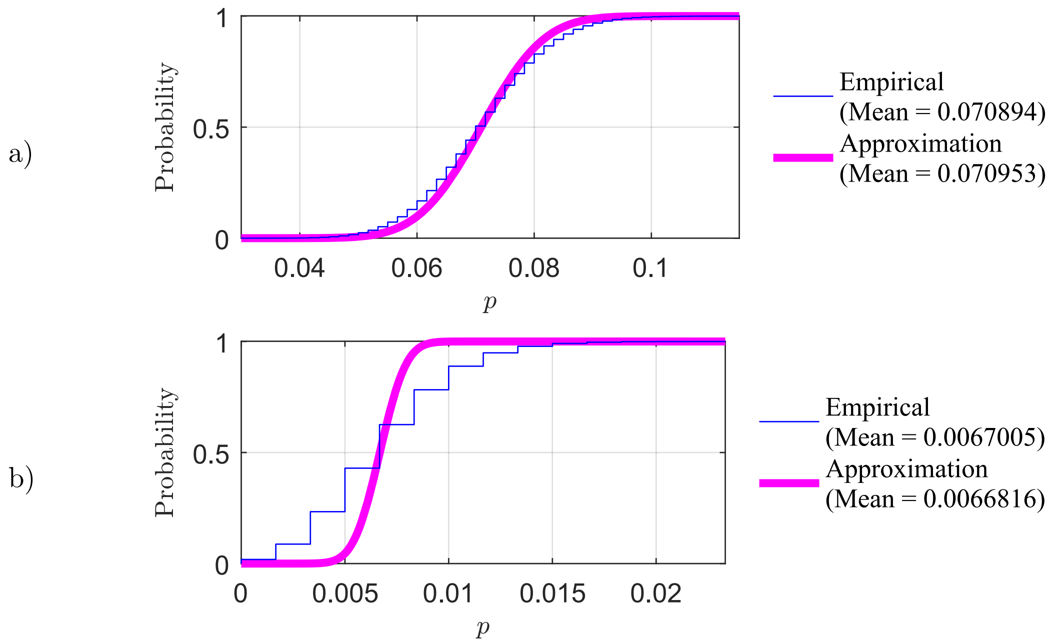

Figure 13: Approximation to distribution of P^k vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$, *times to repair are uniformly distributed on* $[2, 4]$, $z = 150$, $\tau = 600$ *a)* $k = 12$. *b)* $k = 16$.

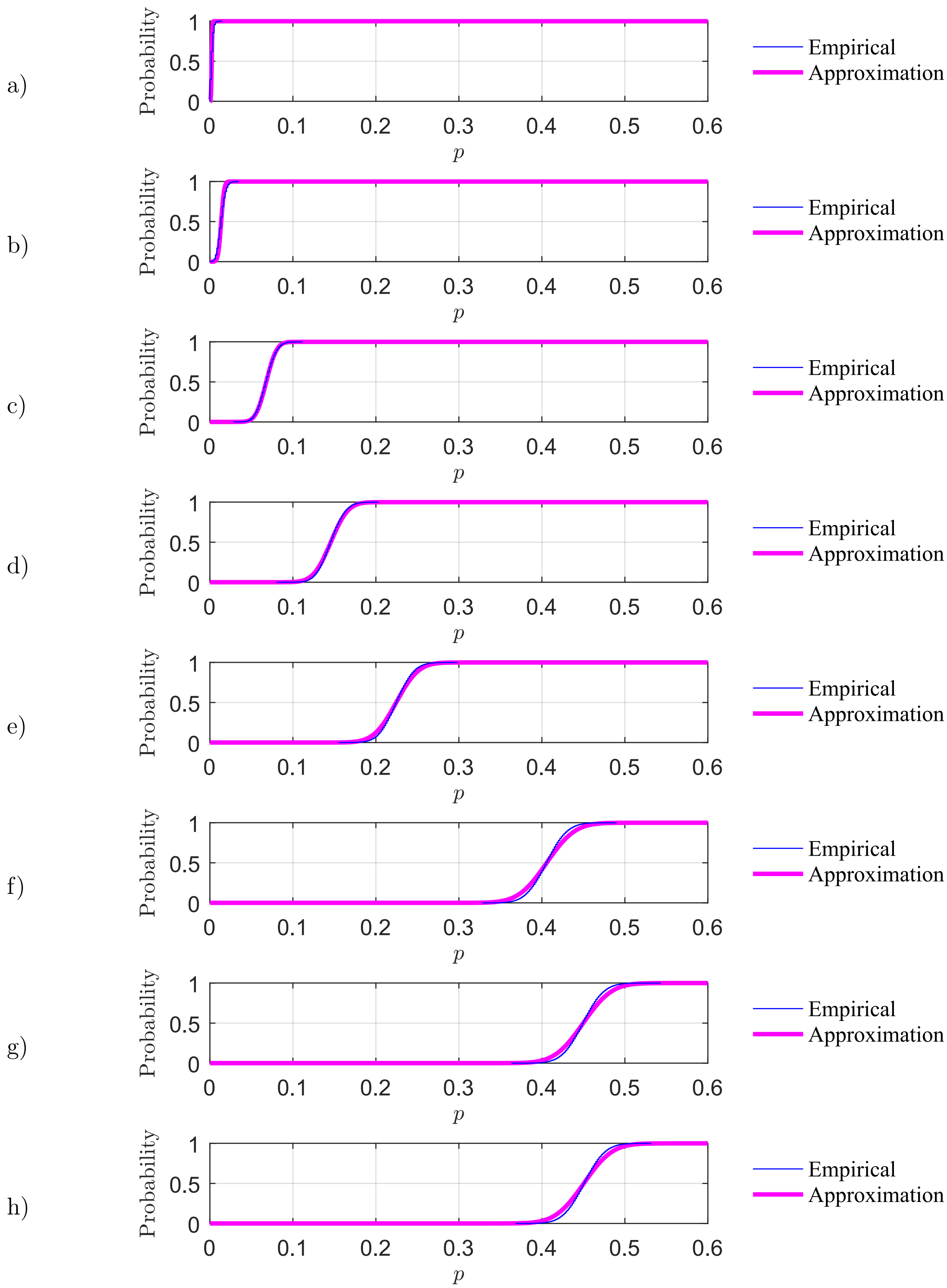

Figure 14: Approximation to distribution of P^3 vs empirical distribution from simulations. Times to fail follow $\ln \mathcal{N}(0.4, 1)$, times to repair are uniformly distributed on $[2, 4]$, $z = 150$, a) $\tau = 50$. b) $\tau = 100$. c) $\tau = 200$. d) $\tau = 300$. e) $\tau = 400$. f) $\tau = 800$. g) $\tau = 1600$. h) $\tau = 6400$.

Proof. Recall that the target is giving away glimpses as a homogeneous Poisson process with rate parameter ζ and let T_k be the time to the k-th arrival. Then T_k follows an Erlang distribution with rate parameter ζ so

$$\mathbb{P}(t \le T_k \le t + \delta t) = \frac{\zeta^k t^{k-1} e^{-\zeta t}}{(k-1)!}$$

But to see the k-th glimpse during $[0, \tau]$ there must be at least k glimpses during that interval. Thus we may also write

$$\begin{aligned}\mathbb{P}(t \le T_k \le t + \delta t) &= \sum_{n=k}^{\infty} \mathbb{P}(t \le T_k \le t + \delta t | n \text{ glimpses in } [0, \tau]) \cdot \mathbb{P}(n \text{ glimpses in } [0, \tau]) \\ &= \sum_{n=k}^{\infty} f_n(t) \frac{\lambda^n e^{-\lambda}}{n!}\end{aligned}$$

where $\sum_{n=k}^{\infty}$ means the infinite series as a sequence of partial sums, $f_n(t) \equiv f_n(t; k, \zeta, \tau)$, and we use $\lambda = \zeta\tau$ as the parameter to a Poisson distribution. Thus

$$\begin{aligned}\frac{\zeta^k t^{k-1} e^{-\zeta t}}{(k-1)!} &= \sum_{n=k}^{\infty} f_n(t) \frac{\lambda^n e^{-\lambda}}{n!} \\ e^{\lambda - \zeta t} &= \sum_{n=k}^{\infty} f_n(t) \frac{(k-1)!}{\zeta^k t^{k-1}} \frac{\lambda^n}{n!} \\ &= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!}{\zeta^k t^{k-1}} \frac{\lambda^{n+k}}{(n+k)!} \\ &= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!}{t^{k-1}} \frac{\lambda^k}{\zeta^k} \frac{n!}{(n+k)!} \frac{\lambda^n}{n!} \\ e^{\lambda(1-\frac{t}{\tau})} &= \sum_{n=0}^{\infty} f_{n+k}(t) \frac{(k-1)!\, n!}{(n+k)!} \frac{\tau^k}{t^{k-1}} \frac{\lambda^n}{n!}\end{aligned}$$

Now by the well-known expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ we have

$$e^{\lambda(1-\frac{t}{\tau})} = \sum_{n=0}^{\infty} \left(1 - \frac{t}{\tau}\right)^n \frac{\lambda^n}{n!}$$

where again $\sum_{n=0}^{\infty}$ means the infinite series as a sequence of partial sums. The two series are equal if and only if their terms are equal. Hence for all positive integers n we have

$$\begin{aligned}f_{n+k}(t) \frac{(k-1)!\, n!}{(n+k)!} \frac{\tau^k}{t^{k-1}} &= \left(1 - \frac{t}{\tau}\right)^n \\ f_{n+k}(t) &= \frac{(n+k)!}{(k-1)!\, n!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^n \\ f_n(t) &= \frac{n!}{(k-1)!\, (n-k)!} \frac{t^{k-1}}{\tau^k} \left(1 - \frac{t}{\tau}\right)^{n-k}\end{aligned}$$

□

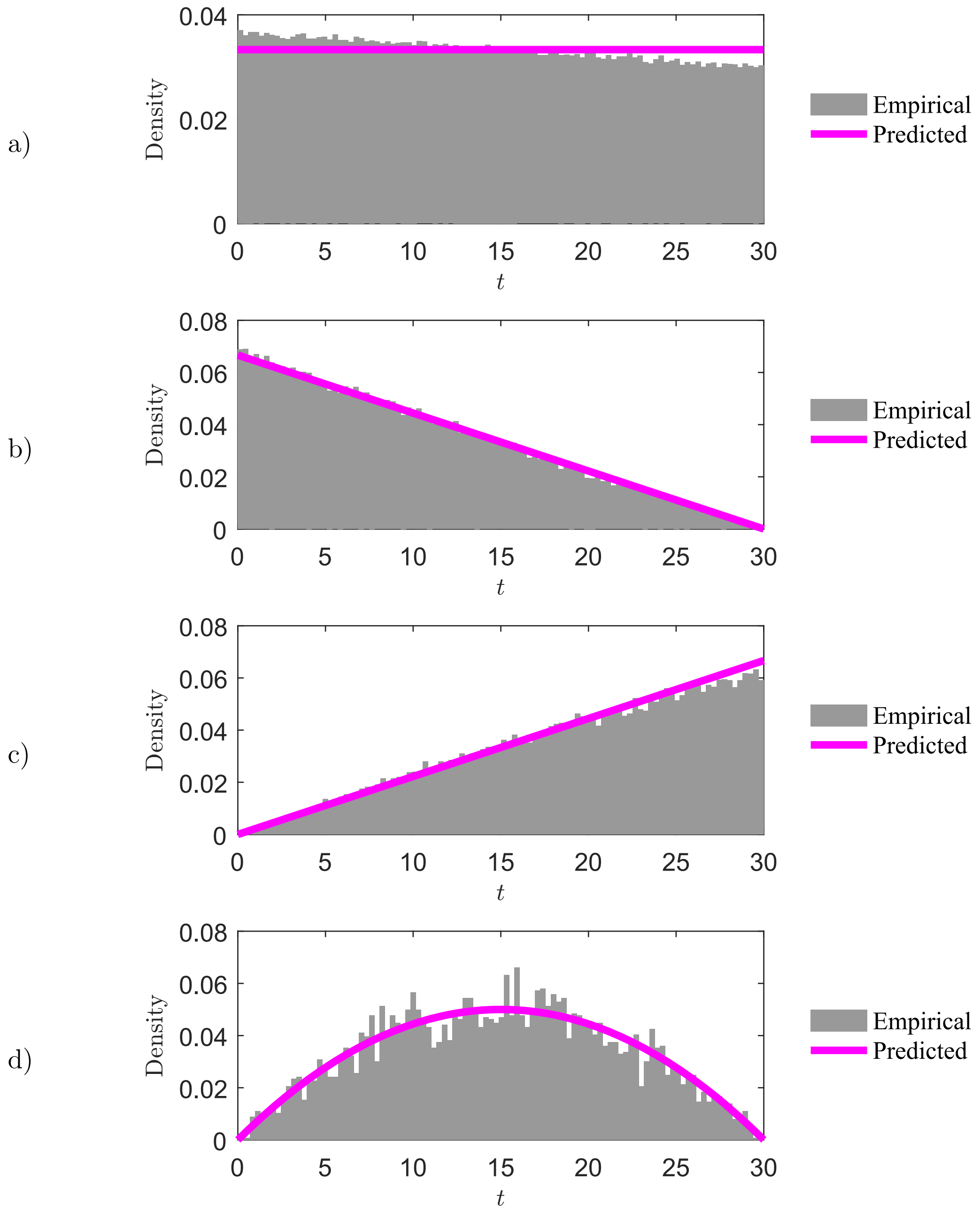

Figure 15: Predicted density of times to k-th glimpse given n glimpses during $[0, \tau]$ vs empirical distribution from simulations. $z = 150$, $\tau = 30$ a) $k = 1, n = 1$. b) $k = 1, n = 2$. c) $k = 2, n = 2$. d) $k = 2, n = 3$.

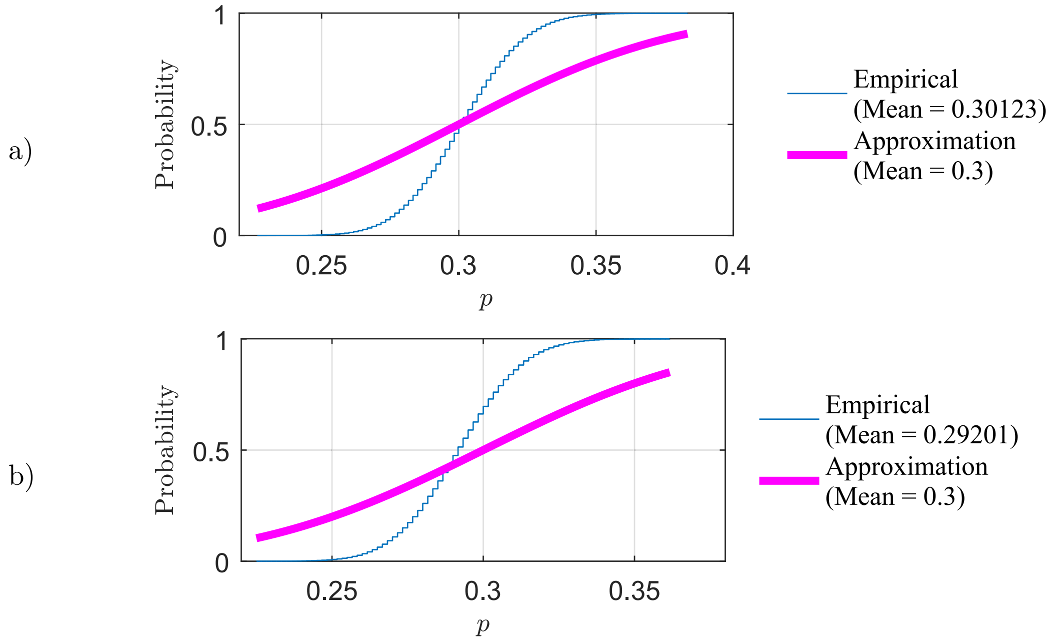

Figure 16: Approximation to distribution of P_n^k *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 30$ *a)* $k = 1, n = 2$*. b)* $k = 2, n = 3$.

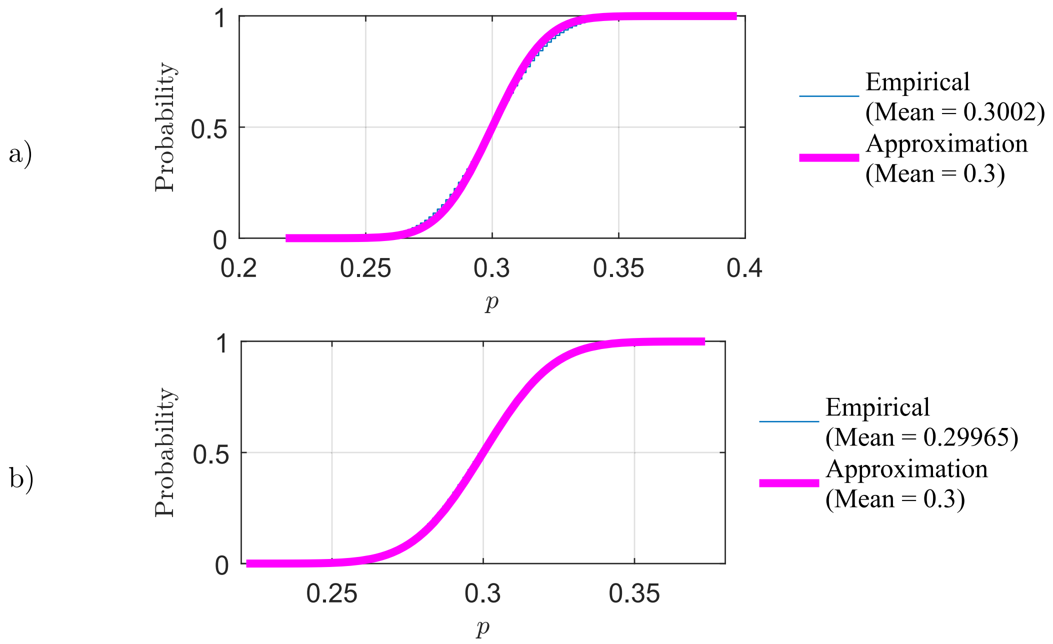

Figure 17: Approximation to distribution of P_n^k *vs empirical distribution from simulations. Times to fail follow* Exp(0.3)*, times to repair follow* Exp(0.7)*,* $z = 150$*,* $\tau = 600$ *a)* $k = 6, n = 9$*. b)* $k = 9, n = 13$.

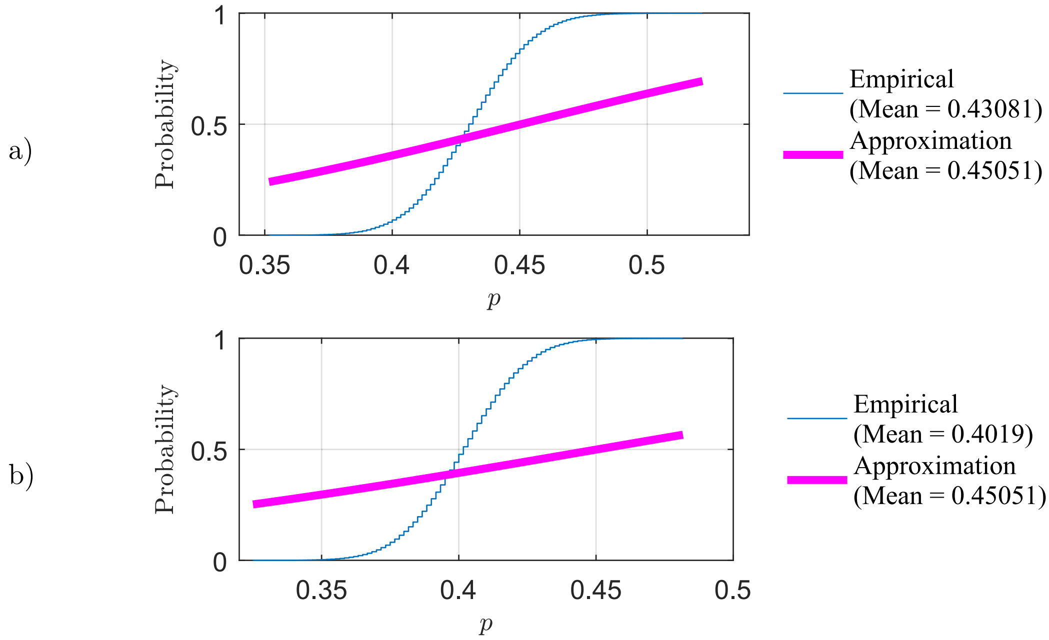

Figure 18: Approximation to distribution of P_n^k *vs empirical distribution from simulations. Times to fail follow* $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*,* $\tau = 30$ *a)* $k = 1, n = 1$*. b)* $k = 1, n = 3$*.*

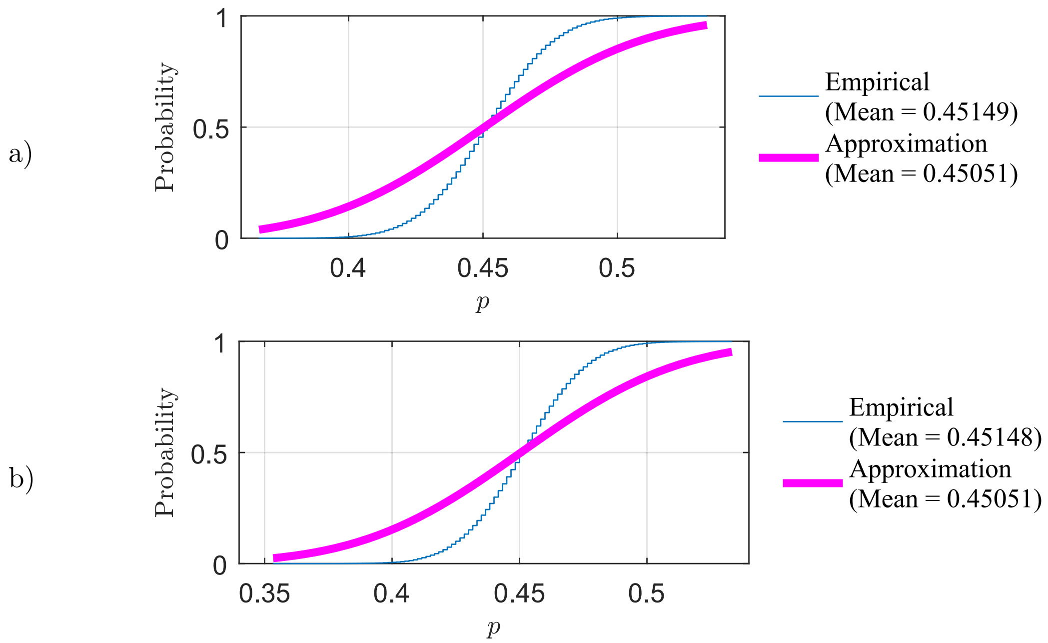

Figure 19: Approximation to distribution of P_n^k *vs empirical distribution from simulations. Times to fail follow* $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*,* $\tau = 600$ *a)* $k = 4, n = 12$*. b)* $k = 11, n = 16$*.*

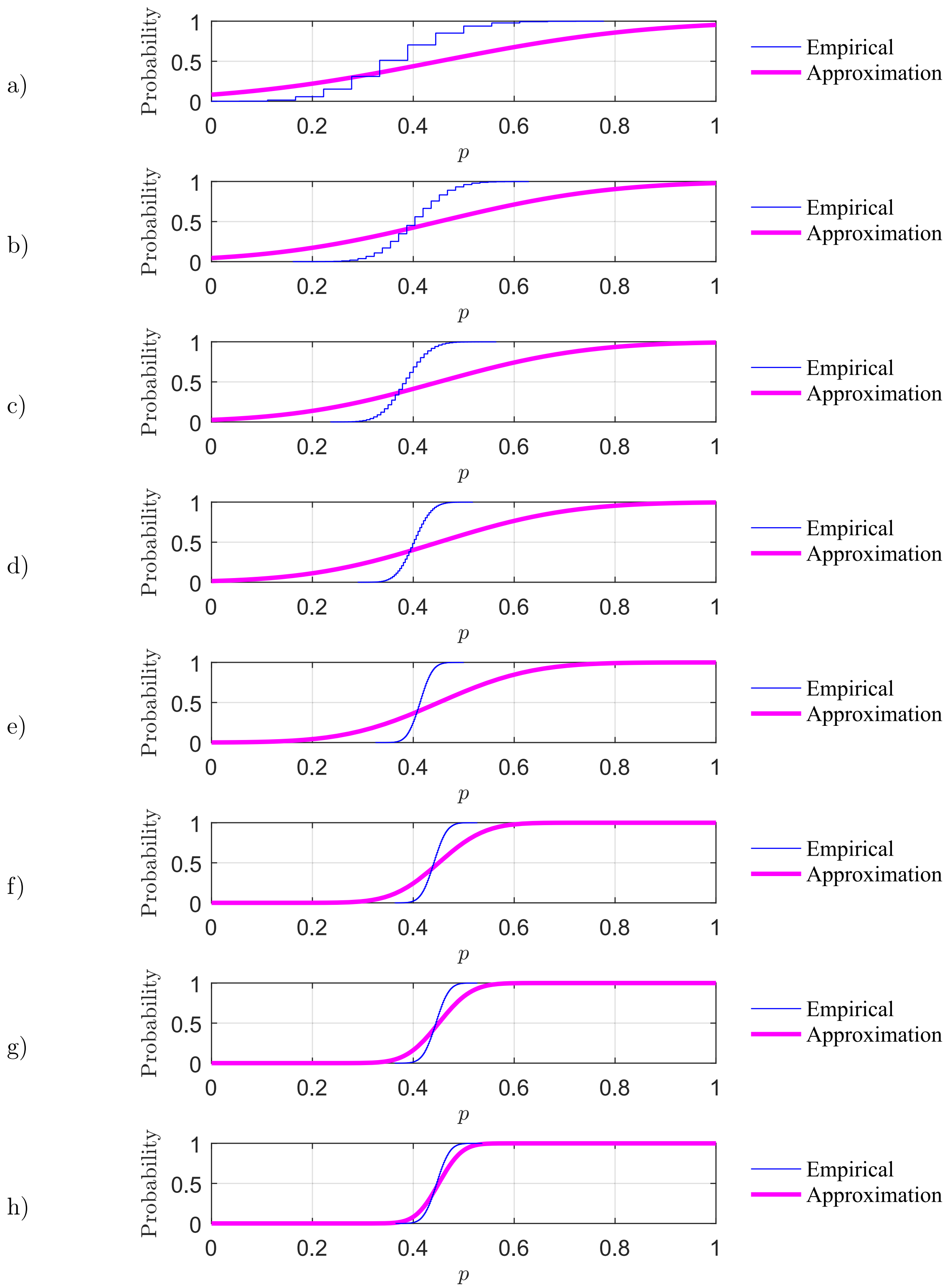

Figure 20: Approximation to distribution of P_3^1 *vs empirical distribution from simulations. Times to fail follow* $\ln \mathcal{N}(0.4, 1)$*, times to repair are uniformly distributed on* $[2, 4]$*,* $z = 150$*, a)* $\tau = 10$*. b)* $\tau = 15$*. c)* $\tau = 20$*. d)* $\tau = 25$*. e)* $\tau = 50$*. f)* $\tau = 200$*. g)* $\tau = 400$*. h)* $\tau = 800$*.*

Proposition 4. To obtain an approximation to P_n^k, apply Lemma 5 with $g(s) = f_n(s\tau; k, \zeta, \tau)$.

Proof. Immediate from Corollary 3. □

Figure 15 checks that the waiting time to the k-th glimpse given n glimpses is indeed distributed in the manner predicted by Lemma 6. Figures 16 through 20 compare Proposition 4's approximation to the distribution of P_n^k with empirical results from simulation. The simulations were conducted as for the previous section. The simulations show agreement between the approximation and empirical values for $\mathbb{E}(P_n^k)$. The agreement between the approximate and empirical distributions is poor when τ is small but improves as $\tau \to \infty$.

5. Conclusion

We studied a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time $[0, \tau]$, the sensor has probability P_n of detecting n glimpses, probability P^k of detecting the k-th glimpse, and probability P_n^k of detecting the k-th glimpse given n glimpses in that interval. The probabilities can provide insight into operations; indeed $1 - P_0$ is the probability of detecting the target, P_n considers the need to see the target multiple times, and P^k and P_n^k could apply in operations that seek to gather a targeted piece of intelligence from an adversary asset.

We devised closed-form approximations to the distributions of P_n, P^k, P_n^k and proved that the approximations become perfect as $\tau \to \infty$ where 'perfect' is formally interpreted as pointwise convergence. Simulations of the equilibrium process showed that for τ finite, the approximations' closeness to reality can range from being poor to good (we did not obtain a guarantee on the rate at which the error decreases as τ increases).

The results can be applied to analysis of operations whenever the intermittent sensor homogenous glimpses model is a valid abstraction of the operation being studied. The means of P_n, P^k, P_n^k could be useful as measures of performance as they can be easily estimated from the approximations that were obtained for those probabilities while being close to the actual values. Moreover we have learned that the distributions of P_n, P^k, P_n^k are each asymptotically 'function-normal', with a normal distribution around some function; indeed P_n is asymptotically log-normal when $n = 0$ and 'Lambert W-normal' when $n \geq 0$, and P^k and P_n^k are both asymptotically normal. This insight can guide the design and verification of simulations when studying the actual distributions of those probabilities.

6. Acknowledgements

The author thanks Stephen Bocquet for his feedback.

7. References

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Appendix A. Proofs of Technical Results

Lemma 1. Let $\{F_\tau\}_\tau$ and $\{G_\tau\}_\tau$ be sequences of cumulative distribution functions, and $\{X_{F_\tau}\}_\tau$ and $\{X_{G_\tau}\}_\tau$ be the corresponding sequences of random variables. Suppose that for all τ, X_{F_τ} and X_{G_τ} are both continuous and non-negative, and $\mathbb{E}(X_{F_\tau})$ and $\mathbb{E}(X_{G_\tau})$ are both finite. Suppose further that for all $x \geq 0$, $\epsilon > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|F_\tau(x) - G_\tau(x)| < \epsilon$. Then for all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.

Proof of Lemma 1. In three steps:

1. *For any $\epsilon_F, \epsilon_G > 0$ there exist $a > 0$, $n > 0$ such that*

$$\left|\mathbb{E}(X_{F_\tau}) - \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right)\right| < \epsilon_F$$

$$\left|\mathbb{E}(X_{G_\tau}) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)\right| < \epsilon_G$$

where $x_k = \frac{k}{n} \cdot a$.

Proof. We have

$$\begin{aligned}
\mathbb{E}(X_{F_\tau}) &= \int_{[0,\infty)} x\, dF_(x)\, dx \\
&= \lim_{a\to\infty} \lim_{n\to\infty} \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) \\
\mathbb{E}(X_{G_\tau}) &= \int_{[0,\infty)} x\, dG_(x)\, dx \\
&= \lim_{a\to\infty} \lim_{n\to\infty} \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)
\end{aligned}$$

Now $\mathbb{E}(X_{F_\tau})$, $\mathbb{E}(X_{G_\tau})$ are finite, so the limits exist and thus a, n exist by definition. □

2. *Let $a, n > 0$ and $x_k = \frac{k}{n} \cdot a$ for $k = 1 \ldots n$. For any $\epsilon_\Sigma > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then*

$$\left|\sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)\right| < \epsilon_\Sigma \qquad (1)$$

Proof. For $k = 1 \ldots n$ put $\epsilon_k = \dfrac{\epsilon_\Sigma}{2n \cdot x_k}$. Then $\epsilon_k > 0$ so by assumption, for each k there exists $\tau'_{1,k}$ such that if $\tau > \tau'_{1,k}$ then $|F_\tau(x_k) - G_\tau(x_k)| < \epsilon_k$ and likewise there exists $\tau'_{2,k}$ such that if $\tau > \tau'_{2,k}$ then $|F_\tau(x_{k-1}) - G_\tau(x_{k-1})| < \epsilon_k$. Set $\tau' = \max\{\max\left(\tau'_{1,k}, \tau'_{2,k}\right) :$

$k = 1 \dots n\}$. Then for any $\tau > \tau'$

$$\begin{aligned}
&\left|\sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)\right| \\
&= \left|\sum_{k=1}^{n} x_k \left(F_\tau(x_k) - G_\tau(x_k)\right) - \sum_{k=1}^{n} x_k \left(F_\tau(x_{k-1}) - G_\tau(x_{k-1})\right)\right| \\
&\leq \sum_{k=1}^{n} x_k \left|F_\tau(x_k) - G_\tau(x_k)\right| + \sum_{k=1}^{n} x_k \left|F_\tau(x_{k-1}) - G_\tau(x_{k-1})\right| \\
&< \sum_{k=1}^{n} x_k \frac{\epsilon_\Sigma}{2n \cdot x_k} + \sum_{k=1}^{n} x_k \frac{\epsilon_\Sigma}{2n \cdot x_k} \\
&= \epsilon_\Sigma
\end{aligned}$$

□

3. *For all $\epsilon' > 0$ there exists $\tau' > 0$ such that if $\tau > \tau'$ then $|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| < \epsilon'$.*

Proof. Choose $\epsilon_\Sigma, \epsilon_F, \epsilon_G > 0$ such that $\epsilon_\Sigma + \epsilon_F + \epsilon_G = \epsilon'$. Construct a, n as per step 1 and then construct τ' via step 2. Then for any $\tau > \tau'$

$$\begin{aligned}
|\mathbb{E}(X_{F_\tau}) - \mathbb{E}(X_{G_\tau})| \leq &\left|\mathbb{E}(X_{F_\tau}) - \sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right)\right| + \\
&\left|\sum_{k=1}^{n} x_k \left(F_\tau(x_k) - F_\tau(x_{k-1})\right) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)\right| + \\
&\left|\mathbb{E}(X_{G_\tau}) - \sum_{k=1}^{n} x_k \left(G_\tau(x_k) - G_\tau(x_{k-1})\right)\right| \\
< &\ \epsilon_F + \epsilon_\Sigma + \epsilon_G
\end{aligned}$$

□

Lemma 3. $\left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right| = \left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right|$.

Proof. Let $Y \sim \mathcal{N}(\mu_U, \sigma_U^2)$ then $-\zeta Y \sim \mathcal{N}(-\zeta\mu_U, \zeta^2\sigma_U^2)$ by properties of the normal distribution. Now $U \leq u$, $Y \leq u$ if and only if $-\zeta U \geq -\zeta u$, $-\zeta Y \geq -\zeta u$ so

$$\begin{aligned}
\left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathcal{N}(-\zeta u; -\zeta\mu_U, \zeta^2\sigma_U^2)\right| &= \left|\mathbb{P}(-\zeta U \leq -\zeta u) - \mathbb{P}(-\zeta Y \leq -\zeta u)\right| \\
&= \left|(1 - \mathbb{P}(-\zeta U \geq -\zeta u)) - (1 - \mathbb{P}(-\zeta Y \geq -\zeta u))\right| \\
&= \left|\mathbb{P}(U \leq u) - \mathbb{P}(Y \leq u)\right| \\
&= \left|\mathbb{P}(U \leq u) - \mathcal{N}(u; \mu_U, \sigma_U^2)\right|
\end{aligned}$$

as required. □

Lemma 7. The function

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

attains its maximum value of

$$\breve{p}_n = e^{-n}\frac{n^n}{n!}$$

when $\lambda = n$, and is increasing if $\lambda < n$ and decreasing if $\lambda > n$.

Proof. We have

$$\begin{aligned} h_n'(\lambda) &= -e^{-\lambda}\frac{\lambda^n}{n!} + e^{-\lambda}\frac{n\lambda^{n-1}}{n!} \\ &= e^{-\lambda}\frac{\lambda^{n-1}}{n!}(n-\lambda) \end{aligned}$$

hence $h_n'(\lambda) < 0$ if $\lambda < n$, $h_n'(\lambda) = 0$ if $\lambda = n$, and $h_n'(\lambda) > 0$ if $\lambda > n$. Thus the point $\lambda = n$ is the global maximum. □

Lemma 8. Given

$$\kappa_n(p) = -\frac{(n!p)^{1/n}}{n}$$

the function

$$h_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}$$

has pre-images

$$\begin{aligned} h_{0,n}^{-1}(p) &= -nW_0(\kappa_n(p)) \\ h_{-1,n}^{-1}(p) &= -nW_{-1}(\kappa_n(p)) \end{aligned}$$

respectively mapping from $[0, \breve{p}_n]$ to $[0, n]$ and from $[0, \breve{p}_n]$ to $[n, \infty)$, where W_0, W_{-1} are the Lambert W function on its 0, -1 branches.

Proof. Suppose

$$p = e^{-\lambda}\frac{\lambda^n}{n!}$$

then

$$\begin{aligned} n!p &= e^{-\lambda}\lambda^n \\ (n!p)^{\frac{1}{n}} &= e^{-\frac{\lambda}{n}}\lambda \\ \kappa_n(p) &= -\tfrac{\lambda}{n}e^{-\frac{\lambda}{n}} \\ -\tfrac{\lambda}{n} &= W(\kappa_n(p)) \\ \lambda &= nW(\kappa_n(p)) \end{aligned}$$

and result follows by applying the two branches of W. □

Lemma 9. The function

$$f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \quad \text{where } t, \lambda \geq 0$$

attains its maximum at $t = \dfrac{k-1}{\lambda}$.

Proof. We have

$$\begin{aligned} f'(t) &= \frac{\lambda^k}{(k-1)!}\left(t^{k-1}(-\lambda e^{-\lambda t}) + (k-1)t^{k-2}e^{-\lambda t}\right) \\ &= \frac{\lambda^k t^{k-2} e^{-\lambda t}}{(k-1)!}\,(k-1-\lambda t) \end{aligned}$$

hence $f'(t) < 0$ if $k-1 < \lambda t$, $f'(t) = 0$ if $k-1 = \lambda t$, and $f'(t) > 0$ if $k-1 > \lambda t$. Thus the point $t = (k-1)/\lambda$ is the global maximum. □

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19. ABSTRACT

This technical note considers a sensor that alternates randomly between working and broken versus a target that reluctantly gives away glimpses as a homogenous Poisson process. Over any interval of time, the sensor has a probability of detecting n glimpses, of detecting the k-th glimpse, and of detecting the k-th glimpse when there are n glimpses in that interval. We devise closed-form approximations to the distributions for those probabilities, prove that the approximations become perfect as the time interval becomes infinitely long (asymptotic distributions, pointwise convergence), and compare the approximations with empirical results obtained from simulations.