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TECHNICAL NOTE

Detailed Complexity Proofs for the Patrol Boat Scheduling Problem with Complete Coverage

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EXECUTIVE SUMMARY

The patrol boat scheduling problem with complete coverage (PBSPCC) is concerned with scheduling a minimum size fleet of resource-constrained vessels to provide ongoing continuous coverage over a set of maritime patrol regions. The problem is complicated by the necessity for patrol vessels to visit replenishment stations (ports) on a regular basis for mandatory resource replenishment. This problem, which is applicable to maritime border protection and surveillance operations, has its origins in fleet sizing questions raised by the Royal Australian Navy (RAN) and has been a subject of interest for the authors since 2008.

While we have focussed on solution techniques for the PBSPCC, most notably integer linear programming (ILP) with column generation based branch-and-bound approaches, the unique nature of the problem, and the fact that it is clearly *not* an instance of, *nor* reducible from, the travelling salesman problem (TSP) or indeed truck scheduling problems, raises the question of how computationally difficult the PBSPCC is. In a recent paper published in *Naval Research Logistics* (NRL), we answered this question and demonstrated that the PBSPCC and its associated computational problems are indeed $\mathcal{NP}$-hard. Specifically we showed:

- The PBSPCC that takes a patrol network and a vessel class, and finds the minimum size fleet of that class which provides complete coverage of that network is $\mathcal{NP}$-hard.
- The PBSPCC decision problem, which takes a patrol network, a fixed fleet of vessels of the same class, and determines whether that fleet can provide complete coverage of the network is $\mathcal{NP}$-hard.
- For a given polynomial p uniformly larger than $3n(n+1)$, the p-bounded cyclic PBSPCC decision problem, which is a variant of the PBSPCC decision problem where the length of cycles in any schedule that provides complete coverage of a patrol network (of size n) is bounded by $p(n)$, is $\mathcal{NP}$-complete.

To meet the editorial requirements for publication, the authors' NRL paper only included descriptive outlines of the main proofs of these assertions. This decision was taken with the view to make the arguments more comprehensible and intuitive for the journal's readership. However, an obvious deficiency in this style of presentation is that it lacks the formalism and rigour usually expected of mathematical proofs. The purpose of this Technical Note, therefore, is to complement the authors' NRL paper by presenting the complexity proofs with full mathematical rigour.

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GLOSSARY

ILP	Integer linear programming
$\mathcal{NP}$	Non-deterministic polynomial-time
NRL	*Naval Research Logistics*
PBSPCC	Patrol boat scheduling problem with complete coverage
RAN	Royal Australian Navy
TSP	Travelling salesman problem
VRP	Vehicle routing problem

NOTATION

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_n$	Set of non-negative integers modulo n
$G = (V, E)$	A graph with vertices V and edges E
π	Path in a graph
P	Set of patrol regions
Q	Set of ports
$N = (V, E, P, Q)$	Patrol network with $P \cup Q = V$ and $P \cap Q = \emptyset$
$\text{size}(N)$	The size of N, equal to $\lvert V\rvert + \lvert E\rvert$
$\mathcal{T}$	Transformation of a graph into a patrol network
T_E	Patrol vessel endurance
T_R	Patrol vessel replenishment time
$d : E \to \mathbb{N}$	Transit time of a vessel along an edge of N
$\ell = (T_E, T_R, d)$	Vessel performance
$\lambda : \mathbb{Z} \to V \cup E$	A route in N
Λ	A collection of feasible routes on N

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1. INTRODUCTION

In 2008 the authors were supporting a patrol vessel acquisition decision for the Royal Australian Navy (RAN). At that time, a key question was how to determine the minimum number of patrol vessels required to maintain a continuous presence in a fixed set of pre-defined patrol regions. To address the client's initial and subsequent requirements, the authors developed a range of approaches to examine this problem. These approaches included simple Excel spreadsheets that estimated the required patrol effort [12], pure linear programming approaches that placed hard lower bounds on the size of any fleet that could meet the continuous presence requirement [12, 14], and ultimately, stochastic simulation models that examined the performance of a fleet [15].

While undertaking this work, the authors became aware of the computational difficulty and uniqueness of the problem. On the one hand, the problem seemed like a set partitioning or assignment problem as all the vessels were supposed to be at each patrol region at each moment in the planning horizon. On the other hand, the problem appeared to possess characteristics resembling a vehicle routing problem (VRP) in that each vessel needed to return to port regularly to refuel, replenish and provide crew layover. The replenishment aspect complicated the problem and made it unlike any problem that the authors were originally able to find in the literature. There were analogues, like patrol car scheduling, where highway patrol vehicles cover accident hotspots at fixed times of the day [8, 9, 10, 1, 5], but at the time there was nothing directly equivalent to the continuous presence requirement. It was only relatively recently that a similar problem emerged, in which, for reasons such as disaster relief, a network of drones is required to provide continuous Wi-Fi coverage to an area [11].

In the years that followed the original client work the authors chose to focus on what became known as the patrol boat scheduling problem with complete coverage (PBSPCC). This is a narrower problem where we consider a homogeneous fleet of vessels over a network of ports and patrol regions and determine the existence and properties of a schedule that provides complete coverage of the patrol regions, either indefinitely or over a fixed time horizon. The PBSPCC became the focus of the second author's PhD research which concluded in 2017 [2] and developed effective column generation techniques to compute such schedules [4, 3]. A key question that arose during this academic study was concerned with the computational complexity of the PBSPCC. Intuitively the problem appeared to be $\mathcal{NP}$-hard, however, demonstrating this was not easy as the problem was not equivalent to, nor reducible from, the travelling salesman problem (TSP) or truck scheduling problems – the usual impetus to return to base for replenishment is constrained by the requirement to be present in each patrol area at each point in time rather than simply visiting a city or making a delivery to a customer.

Recently, the authors were able to show that the PBSPCC is indeed $\mathcal{NP}$-hard by providing a transformation to/from the Hamiltonian graph decision problem, and published this work in *Naval Research Logistics* (NRL) [13]. Specifically, the authors showed that:

- The PBSPCC that takes a patrol network and a vessel class, and finds the minimum size fleet of that class which provides complete coverage of that network is $\mathcal{NP}$-hard.

- The PBSPCC decision problem, which takes a patrol network, a fixed fleet of vessels of the same class, and determines whether that fleet can provide complete coverage of the network is $\mathcal{NP}$-hard.

- For a given polynomial p uniformly larger than $3n(n+1)$, the p-Bounded Cyclic PBSPCC decision problem, which is a variant of the PBSPCC decision problem where the length of cycles in any schedule that provides complete coverage of a patrol network (of size n) is bounded by $p(n)$, is $\mathcal{NP}$-complete.

To meet the editorial requirements for publication, the NRL paper [13] only included descriptive outlines of the main proofs of these assertions. This decision was taken with the view to make the arguments more comprehensible and intuitive for the journal's readership. However, an obvious deficiency in this style of presentation is that it lacks the formalism and rigour usually expected of mathematical proofs. The purpose of this Technical Note, therefore, is to complement the authors' NRL paper by presenting the complexity proofs with full mathematical rigour.

This technical note should be read in conjunction with the authors' NRL paper [13]. We will take the introductory, motivational and explanatory material presented as read, and include only the main definitions and subsequent results and proofs, which we present in full.

2. PRELIMINARIES

For a graph $G = (V, E)$, we use the definitions of a *Hamiltonian circuit*, whether the graph is *Hamiltonian*, and the *Hamiltonian graph decision problem* (GT37 in [6]) as per Definitions 5.1 and 5.2 of [13] and note that [7] shows that this is $\mathcal{NP}$-complete.

For a graph $G = (V, E)$, we use the definitions of a *path*, *patrol network* $N = (V, E, P, Q)$, P *patrol regions*, Q *ports*, $\ell = (T_E, T_R, d)$ *vessel performance*, T_E *endurance*, T_R replenishment time, $d : E \to \mathbb{N}$ *transit time*, (N, ℓ) *patrolled network*, the *passage at time* t, ℓ*-feasible*, a collection Λ *covers* (N, ℓ), the *size* of a complete cover, *cyclic* routes and the *order* of those cyclic routes as in Definitions 4.1 through 4.12 of [13].

For clarity we will repeat the definitions of the three computational problems revisited in this technical note (refer to Definitions 4.10 and 4.13 in [13]):

Definition 2.1. *The* ***patrol boat scheduling problem with complete coverage (PBSPCC)*** *is a computational problem which takes as input (N, ℓ), a patrolled network, and finds a cover of minimum cardinality.*

Definition 2.2. *The* ***PBSPCC decision problem*** *is a computational problem which takes as input (N, ℓ) a patrolled network and a $m \in \mathbb{N}$ and determines whether (N, ℓ) has a complete cover of cardinality m.*

Definition 2.3. *Let $p : \mathbb{N} \to \mathbb{N}$ be an increasing polynomial function. The* ***p-bounded cyclic PBSPCC decision problem*** *is a computational problem which takes as input a patrolled network (N, ℓ), an integer m and determines whether there is a patrol cover of (N, ℓ) of size m, which is cyclic of order k for some $k \leq p(\mathrm{size}(N))$, where $\mathrm{size}(N) = |V| + |E|$.*

When exploring a full formal proof, the concept of a *route* in a patrol network (Definition 4.5 in [13]) is a key definition, and an activity at a point in a route is key to the subsequent technical propositions, so we repeat the definition here:

Definition 2.4. *Let $N = (V, E, P, Q)$ be a patrol network. We call $\lambda : \mathbb{Z} \to V \cup E$ a* ***route*** *in N if and only if, for all $t \in \mathbb{Z}$, one of the following holds:*

1. $\lambda(t) \in \lambda(t+1) \in E$,

2. $\lambda(t+1) \in \lambda(t) \in E$,

3. $\lambda(t), \lambda(t+1) \in E$ *and* $\lambda(t) \cap \lambda(t+1) \neq \emptyset$, *or*

4. $\lambda(t+1) = \lambda(t)$.

We will now make two elementary observations about how routes in patrol networks behave.

Proposition 2.1. *Let λ be a route in a patrol network N and suppose that we have $t \in \mathbb{Z}$ such that $\lambda(t) \in V$. Then:*

1. $\lambda(t+1) = \lambda(t)$ *or* $\lambda(t+1) \in E$, *and*

2. $\lambda(t-1) = \lambda(t)$ *or* $\lambda(t-1) \in E$.

Proof. Under the hypotheses of this proposition, only conditions 1. and 4. from Definition 2.4 hold in the definition of a route, giving us Item 1 of Proposition 2.1. However, if we apply Definition 2.4 at time $t-1$, we see that only conditions 2. and 4. hold, giving us Item 2 of Proposition 2.1. □

Proposition 2.2. *Let λ be a route in a patrol network N. Then the following two statements hold:*

$$(\forall t \in \mathbb{Z})\,[\lambda(t) \in P \implies \lambda(t+1) \notin Q \text{ and } \lambda(t) \notin Q \text{ and } \lambda(t-1) \notin Q]\,,$$
$$(\forall t \in \mathbb{Z})\,[\lambda(t) \in Q \implies \lambda(t+1) \notin P \text{ and } \lambda(t) \notin P \text{ and } \lambda(t-1) \notin P]\,.$$

Proof. Let $t \in \mathbb{Z}$ and suppose that $\lambda(t) \in P$. Since P and Q are disjoint sets, then we have $\lambda(t) \notin Q$. Also, by Proposition 2.1, $\lambda(t+1) = \lambda(t) \in P$ or $\lambda(t+1) \in E$. In both cases this shows that $\lambda(t+1) \notin Q$. Similarly $\lambda(t-1) = \lambda(t)$ or $\lambda(t-1) \in E$, so in both cases $\lambda(t-1) \notin Q$. A symmetrical argument shows that the second statement of the proposition also holds. □

We use the transformation $\mathcal{T}(\cdot)$ from Section 5 of [13] which takes a graph $G = (V, E)$ and expands it into a patrol network $N' = (V', E', P, Q)$, where $P = V$, $Q = E$, $V' = P \cup Q$ and:

$$E' = \{\{x, e\} \mid x \in e \in E\}\,,$$

and a patrol problem (N', ℓ), where $\ell = (3n+2, 1, d)$ and d is uniformly 1 on E'. Figure 1 is an illustration of this transformation.

We conclude this section by noting that our definition of ℓ-feasible routes in a patrol network N' implies that a vessel will visit a port in any time window of length $3(n+1)$ time units.

Proposition 2.3. *For λ, an ℓ-feasible route in N', and $t \in \mathbb{Z}$, there exists a $t' \in \mathbb{Z}$ such that $t < t' \leq t + 3(n+1)$, and $\lambda(t') \in Q$.*

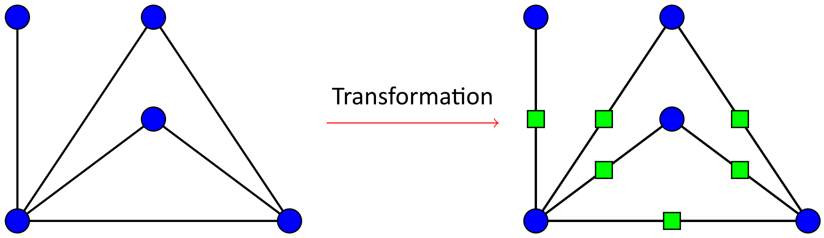

Figure 1 A simple graph G (left) and its transformation N' (right). Patrol regions are represented by solid circles, and the ports are represented by squares.

Proof. Consider the time since a replenishment of length 1 at time $t + 3(n+1)$:

$$k := \min\{t + 3(n+1) - j \mid j \in \mathbb{Z}, j \leq t + 3(n+1) \text{ and } \lambda(j) \in Q\}.$$

Since λ is ℓ-feasible, $k \leq 3n + 2$. Let t' witness that k is the minimum. Therefore:

- $t + 3(n+1) - t' = k \leq 3n + 2$, so $t + 1 \leq t'$, and therefore $t < t'$,
- $t' \leq t + 3(n+1)$, and
- $\lambda(t') \in Q$, as required.

□

3. EQUIVALENCE

In this section we revisit the two main lemmata of [13] that had both been substantially abridged during the peer review process preceding that paper's publication in *Naval Research Logistics*. These lemmata show that for a graph $G = (V, E)$ with $|V| = n$:

- If G is Hamiltonian then $\mathcal{T}(G)$ has a complete cover of size $n+1$ (Lemma 3.1 below and Lemma 5.1 in [13]).
- If $\mathcal{T}(G)$ has a cover of size $n + 1$ then this naturally gives rise to a Hamiltonian circuit on G (Lemma 3.2 below and Lemma 5.2 in [13]).

Lemma 3.1. *If $G = (V, E)$ is a Hamiltonian graph with $|V| = n \geq 2$, then $\mathcal{T}(G)$ has a patrol cover of size $n + 1$ which is cyclic of order at most $3n(n + 1)$.*

Proof. Let $\pi : \mathbb{Z}_n \to V$ define a path such that the sequence:

$$\langle \pi(0), \pi(1), \ldots, \pi(n-1), \pi(0) \rangle$$

forms a Hamiltonian circuit on G and therefore witnesses that G is Hamiltonian. We may extend π to a function $\pi : \mathbb{Z} \to V$ by setting $\pi(m) = \pi(m \bmod n)$. Because π gives the Hamiltonian circuit, π is surjective (onto) and

$$(\forall m \in \mathbb{Z}) \, [\{\pi(m), \pi(m+1)\} \in E] \, .$$

Define $x : \mathbb{Z} \to Q$ by

$$x(m) := \{\pi(m), \pi(m+1)\} \, .$$

That is, $x(m)$ is the port in $\mathcal{T}(G)$ between $\pi(m)$ and $\pi(m + 1)$. We then define $w, y : \mathbb{Z} \to E'$ as follows:

$$w(m) := \{\pi(m), x(m)\} \, ,$$
$$y(m) := \{x(m), \pi(m+1)\} \, .$$

The layout of w, x, and y is given in Figure 2.

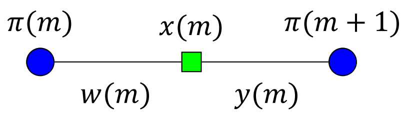

Figure 2 Illustration of defined values $w(m)$, $x(m)$ and $y(m)$.

Thus, $\langle \pi(m), w(m), x(m), y(m), \pi(m+1) \rangle$ is a partial route in N', that is, $\pi(m) \in w(m)$, $x(m) \in w(m)$, $x(m) \in y(m)$ and $\pi(m+1) \in y(m)$, therefore satisfying the conditions imposed on a route at each element of the sequence. Note also that because π cycles, w, x, and y cycle as well, that is, for all $m \in \mathbb{Z}$ we have $w(m) = w(m \bmod n)$, $x(m) = x(m \bmod n)$, and $y(m) = y(m \bmod n)$.

We will now string these partial routes together to make a ℓ-feasible route λ with periodicity $3n(n+1)$, where a vessel stays at each patrol region $\pi(m)$ for $3n$ time units and spends the subsequent 3 time units transiting to the next node in the Hamiltonian circuit.

Define $c : \mathbb{Z} \to \mathbb{Z}$ and $\delta : \mathbb{Z} \to \mathbb{Z}_{3(n+1)}$ by:

$$c(t) := \left\lfloor \frac{t}{3(n+1)} \right\rfloor,$$
$$\delta(t) := t \bmod 3(n+1).$$

Therefore, for all $t \in \mathbb{Z}$, we have $t = 3(n+1)c(t) + \delta(t)$, and if $t+1$ is not a multiple of $3(n+1)$, we have $\delta(t+1) = \delta(t) + 1$. We require each route λ that we are constructing to move on to a new patrol region every $3(n+1)$ time units. Thus, we have two cycles at work: the larger cycle of moving around the patrol network and the shorter cycle of what happens at a patrol region and its subsequent transition or replenishment. The function c will count larger cycles, giving us the node in the Hamiltonian cycle, and δ will count where we are in the smaller cycle of patrolling and moving on to the next patrol region.

Formally we define $\lambda : \mathbb{Z} \to V' \cup E'$ as follows:

$$\lambda(t) = \begin{cases} \pi(c(t)) & \text{if } \delta(t) < 3n, \\ w(c(t)) & \text{if } \delta(t) = 3n, \\ x(c(t)) & \text{if } \delta(t) = 3n+1, \\ y(c(t)) & \text{if } \delta(t) = 3n+2. \end{cases}$$

To see that λ is a route, we note that λ is constant on the first $3n$ time units in each smaller cycle, that is, $\lambda(t) = \pi(c(t)) = \pi(m)$ for some $m \in \mathbb{Z}$ if $\delta(t) < 3n$. Now suppose that $\delta(t) = 3n - 1$. Then $c(t+i) = c(t) = m$ for $i = 1, 2, 3$, and $c(t+4) = c(t) + 1 = m + 1$. Thus at time points $\langle t, t+1, t+2, t+3, t+4 \rangle$, λ moves through the following:

$$\langle \pi(m), w(m), x(m), y(m), \pi(m+1) \rangle,$$

which, as we saw, was constructed to be a partial route in N'. The route λ has exactly 1 time unit on each transition edge, which is defined to have a distance of 1 unit and matches the defined vessel speed of 1 distance unit per time unit.

To see that λ is ℓ-feasible, note that it goes into port $x(c(t))$ every time $\delta(t) = 3n + 1$, and this happens exactly once (for one time unit) in every $3(n+1)$-long cycle. Hence, it is spending $3n + 2$ time units away from port where it had a replenishment of length 1, which is exactly the defined vessel endurance.

We note that λ satisfies the following property:

$$(\forall t, k \in \mathbb{Z})\,[\lambda(t) = \lambda(t + 3n(n+1)k)]\,,$$

that is, λ is cyclic of order $3n(n+1)$. This is because π is cycling with a period n and c is defined to increment every $3(n+1)$ time steps so together they repeat with a period of $3n(n+1)$.

Now, for $i \in \mathbb{Z}$, we define $\lambda_i : \mathbb{Z} \to V' \cup E'$ by $\lambda_i(t) = \lambda(t + 3ni)$. Then the set $\Lambda := \{\lambda_i \mid i \in \mathbb{Z}\}$ has exactly $(n+1)$ members because for all $t \in \mathbb{Z}$:

$$\begin{aligned}\lambda_{i+(n+1)k}(t) &= \lambda(t + 3n(i + (n+1)k)),\\ &= \lambda((t + 3ni) + 3n(n+1)k),\\ &= \lambda(t + 3ni),\\ &= \lambda_i(t).\end{aligned}$$

Therefore $\lambda_i = \lambda_{i+(n+1)k}$. In addition, the members of Λ are cyclic of order $3n(n+1)$. Since λ is ℓ-feasible, all the λ_i in Λ are ℓ-feasible. Hence, we will have shown that $\mathcal{T}(G)$ can be covered by $n + 1 = |V| + 1$ vessels if we can show that Λ hits every node in $P = V$ at every time point.

Let $t \in \mathbb{Z}$ and $v \in P$ (where $P = V$ and $|V| = n$). Since π defines a Hamiltonian circuit on $G = (V, E)$, there is an $m \in \mathbb{Z}$ such that $\pi(m) = v$. Let

$$\begin{aligned}i &= \left\lceil \frac{3(n+1)m - t}{3n} \right\rceil,\\ \delta_0 &= 3ni - (3(n+1)m - t).\end{aligned}$$

Thus $0 \leq \delta_0 < 3n$, and

$$t + 3ni = 3(n+1)m + \delta_0, \quad \text{with } 0 \leq \delta_0 < 3n.$$

So $\lambda_i(t) = \lambda(t + 3ni)$, and we claim that $\lambda(t + 3ni) = \pi(m) = v$. To calculate the value of $\lambda(t + 3ni)$ note that:

$$
\begin{aligned}
c(t+3ni) &= \left\lfloor \frac{t+3ni}{3(n+1)} \right\rfloor, \\
&= \left\lfloor \frac{3(n+1)m+\delta_0}{3(n+1)} \right\rfloor, \\
&= m,
\end{aligned}
$$

since $\delta_0 < 3n < 3(n+1)$, and $\delta(t+3ni) = \delta_0 < 3n$. Thus:

$$
\begin{aligned}
\lambda_i(t) &= \lambda(t+3ni), \\
&= \pi(c(t+3ni)), \\
&= \pi(m), \\
&= v.
\end{aligned}
$$

□

Lemma 3.2. *If $G = (V, E)$ is a graph with $|V| = n \geq 2$ and $\mathcal{T}(G)$ has a patrol cover of size $n+1$, then G is Hamiltonian.*

Proof. Let N' be the patrol network obtained by applying the transformation $\mathcal{T}(G)$. As per the hypotheses of the lemma, assume the patrol network (N', ℓ) has a patrol cover of size $n+1$ and let Λ be that cover. Now N' has $|P| = |V| = n$ patrol regions and at any one time $t \in \mathbb{Z}$, we have $P \subseteq \{\lambda(t) \mid \lambda \in \Lambda\}$. Therefore, at time $t \in \mathbb{Z}$, we have:

$$
|\{\lambda(t) \mid \lambda \in \Lambda, \lambda(t) \notin P\}| \leq 1.
$$

That is, at most one $\lambda \in \Lambda$ can have $\lambda(t) \notin P$, or in patrol operations terminology, only one vessel can be not patrolling at any one time.

If we look at patrol activity, at any one time we will see at most one vessel doing something other than patrolling. We use this observation to demonstrate that if we monitor one non-patrolling vessel at a time and follow the changes from one non-patrolling vessel to another, the aggregate behaviour will be that of a **virtual vessel**, which traces out a Hamiltonian circuit in G. To do this, we need to show that there must be at least one vessel not patrolling at any one time, and that when we track the positions of each non-patrolling vessel, these will join to form a connected path in the graph.

The difficulty with this approach is that (for all we know) the virtual vessel could bounce around going back-and-forth along a non-cyclic path in the network; it could pause, it could keep going back to the same port for replenishment, or it could pass over ports, skipping replenishment altogether.

Noting this possibility, we present a proof by contradiction. We will show that if any virtual vessel fails to stick to the specified program (transit through to a port, replenish, transit to the next node and then keep going), there will be an accruing deficit, which, at the end of $3(n+1)$ time steps, will entail that one vessel has exceeded its endurance threshold.

To uncover the aforementioned deficit, we construct three functions recursively:

$$\alpha : \mathbb{N} \to \mathbb{Z},$$
$$\beta : \mathbb{N} \to Q,$$
$$\gamma : \mathbb{N} \to \Lambda.$$

The function α gives the sequence of times when any vessel goes into replenishment, the function β gives the replenishment port at which that happens, and γ will give the vessel that is going into that port. Note that, in this case, all other vessels are on station in the patrol regions. As we define these functions, we will simultaneously prove that the following condition holds at each $k \in \mathbb{N}$:

$$(\forall t \in \mathbb{Z})(\forall \lambda \in \Lambda)\,[(\lambda \neq \gamma(k) \text{ and } \alpha(k) - 2 \leq t < \alpha(k+1)) \implies \lambda(t) \notin Q]\,.$$

We will call this the **exclusivity condition** at $k \in \mathbb{N}$. We will also simultaneously prove that for $k \in \mathbb{N}$, the following conditions are satisfied:

- $\beta(k) = \gamma(k)(\alpha(k))$,
- $\gamma(k) \neq \gamma(k+1)$,
- $\alpha(k+1) \geq \alpha(k) + 3$.

The set Λ is non-empty, so take λ_0 to be an arbitrary element of Λ. We know from Proposition 2.3 that there is an $t_0 \in \mathbb{Z}$ with $0 < t_0 \leq 3(n+1)$ with $\lambda_0(t_0) \in Q$. Now let $\alpha(1) = t_0$, $\gamma(1) = \lambda_0$, and $\beta(1) = \gamma(1)(\alpha(1))$. Suppose that for some $k \in \mathbb{N}$ we have defined $\alpha(k)$, $\beta(k)$ and $\gamma(k)$ such that $\beta(k) = \gamma(k)(\alpha(k))$. Let

$$\alpha(k+1) = \min\{t \in \mathbb{Z} \mid t > \alpha(k) \text{ and } (\exists \lambda \in \Lambda)\,[\lambda \neq \gamma(k) \text{ and } \lambda(t) \in Q]\}\,. \tag{1}$$

We note that such a minimum exists because Λ has size $n+1$ (which is at least 3), and Proposition 2.3 guarantees that any $\lambda \in \Lambda$ will eventually and repeatedly hit Q.

Let $\gamma(k+1)$ be the element of Λ that witnesses $\alpha(k+1)$ is the minimum described in (1), that is, $\gamma(k+1)$ takes the role of λ in (1). Then immediately we have that $\gamma(k+1) \neq \gamma(k)$ and we can define $\beta(k+1) = \gamma(k+1)(\alpha(k+1))$. Since $\gamma(k)(\alpha(k)) \in Q$, Proposition 2.2 tells us that $\gamma(k)(\alpha(k)+1) \notin P$, so since at most one vessel is not patrolling at any one time, we know that

$\gamma(k+1)(\alpha(k)+1) \in P$. Then Proposition 2.2 tells us that $\gamma(k+1)(\alpha(k)+2) \notin Q$. Using this fact, and given that $\alpha(k+1)$ is minimal with $\gamma(k+1)(\alpha(k+1)) \in Q$, it follows that $\alpha(k+1) \geq \alpha(k)+3$.

We now only need to prove that the exclusivity condition holds at $k \in \mathbb{N}$. Therefore, let $t \in \mathbb{Z}$, $\lambda \in \Lambda$ be such that $\lambda \neq \gamma(k)$ and $\alpha(k) - 2 \leq t < \alpha(k+1)$. We must show that $\lambda(t) \notin Q$. Let us examine the following cases.

- **Case 1.** $t > \alpha(k)$. In this case we immediately know that $\lambda(t) \notin Q$, otherwise t and λ would witness that $\alpha(k+1)$ is not the minimum it is defined to be.
- **Case 2.** $t \in \{\alpha(k)-2, \alpha(k)-1, \alpha(k)\}$. We know that $\gamma(k)(\alpha(k)) = \beta(k) \in Q$, so by Proposition 2.2, $\gamma(k)(\alpha(k)-1) \notin P$. Thus as only one vessel can be not patrolling at one time we know that $\lambda(\alpha(k)-1) \in P$. Proposition 2.2 then tells us that $\lambda(t) \notin Q$ for all three possibilities of t in this case.

Thus, we have defined α, β and γ satisfying our desired properties.

Claim 1. $(\forall k \in \mathbb{N})\,[\alpha(k+1) = \alpha(k)+3]$.

Proof. Assume the contrary. Thus, there is a $k_0 \in \mathbb{N}$ such that $\alpha(k_0+1) \neq \alpha(k_0)+3$, but since we know that $\alpha(k_0+1) \geq \alpha(k_0)+3$, we have that $\alpha(k_0+1) > \alpha(k_0)+3$. Let $\lambda \in \Lambda \setminus \{\gamma(k_0+i) \mid i \in \mathbb{Z}_n\}$. Note that this set is not empty because Λ has size $n+1$. Now by the exclusivity condition holding at $k_0, k_0+1, \ldots, k_0+n-1$, we have:

$$(\forall t \in \mathbb{Z})\,[\alpha(k_0) - 2 \leq t < \alpha(k_0+n) \implies \lambda(t) \notin Q].$$

However, for all $k \in \mathbb{N}$, $\alpha(k+1) \geq \alpha(k)+3$, and combining this with our assumption, we have $\alpha(k_0+n) > \alpha(k_0)+3n$. So we can conclude that:

$$(\forall t \in \mathbb{Z})\,[\alpha(k_0) - 2 \leq t \leq \alpha(k_0)+3n \implies \lambda(t) \notin Q].$$

Thus, at time $\alpha(k_0)+3n$, λ's time since replenishment is at least $\alpha(k_0)+3n-(\alpha(k_0)-3) = 3n+3 = 3(n+1)$. But λ's endurance is at most $3n+2$, so this is a contradiction to λ being ℓ-feasible. □

Claim 2. *For all* $k \in \mathbb{N}$*:*

1. $\gamma(k)(\alpha(k)) \in Q$,
2. $\gamma(k)(\alpha(k)+1) \in E'$,

3. $\gamma(k)(\alpha(k)+2) = \gamma(k+1)(\alpha(k)+1) \in P$,

4. $\gamma(k+1)(\alpha(k)+2) \in E'$,

5. $\gamma(k+1)(\alpha(k)+3) \in Q$.

Proof. Let $k \in \mathbb{N}$. We already know that 1. and 5. hold as we constructed γ and α that way (noting that $\alpha(k+1) = \alpha(k)+3$). The other three items are obtained by noting that $\gamma(k)$ and $\gamma(k+1)$ are distinct vessels of which only one cannot be patrolling at a time, and so they must "swap out" at some time between $\alpha(k)$ and $\alpha(k)+3$. Given our construction of the patrol network, $\gamma(k)$ must transit to a patrol region at time $\alpha(k)+1$ (giving us 2.), to take over patrolling at time $\alpha(k)+2$ from $\gamma(k+1)$, which was patrolling at time $\alpha(k)+1$ (giving us 3.). Then $\gamma(k+1)$ must quickly proceed to be at a port at time $\alpha(k)+3$, so it must undertake a transit at time $\alpha(k)+2$ (which gives us 4.). □

Therefore, for any $k \in \mathbb{N}$, $\gamma(k)(\alpha(k)+2) \in P$ is connected to $\gamma(k+1)(\alpha(k+1)+2) \in P$ in G via the edge $\gamma(k+1)(\alpha(k)+3) \in Q$. So if we define $\rho : \mathbb{N} \to P$ by:

$$\rho(k) := \gamma(k)(\alpha(k)+2),$$

we get a path in G on the nodes of V (as we defined $P = V$).

Claim 3. $(\forall v \in P)\,(\forall k \in \mathbb{N})\,(\exists i < n)\,[\rho(k+i) = v]$.

Proof. Let $v \in P$, and $k \in \mathbb{N}$. Suppose not, that is:

$$(\forall i < n)\,[\rho(k+i) \neq v].$$

Let $\lambda \in \Lambda$ be such that $\lambda(\alpha(k)) = v$, which must exist because each patrol region is covered. We will show by induction that:

$$(\forall i \leq n)\,(\forall t \in \mathbb{Z})\,[\alpha(k)-1 \leq t \leq \alpha(k+i)+1 \implies \lambda(t) = v]. \tag{2}$$

For the base case, that is, $i = 0$, we must show that $\lambda(t) = v$ for $t \in \{\alpha(k)-1, \alpha(k), \alpha(k)+1\}$. Here, we note that $\gamma(k)(\alpha(k)) \in Q$, so $\lambda \neq \gamma(k)$ and by Proposition 2.2 we see that $\gamma(k)(t) \notin P$ for $t \in \{\alpha(k)-1, \alpha(k), \alpha(k)+1\}$. This implies that λ is stuck on a patrol region and therefore constant at v on that same set.

Suppose that (2) holds for some i, where $0 \le i < n$. We must show that it holds for $i+1$. We know by the inductive hypothesis that for $t \in \mathbb{Z}$ such that $\alpha(k) - 1 \le t \le \alpha(k+i)$, we get $\lambda(t) = v$. Let us now follow through the remaining time steps.

- At time $t = \alpha(k+i) + 1$, $\gamma(k+i)(t) \in E'$, so λ must stay at v.
- At time $t = \alpha(k+i) + 2$, $\gamma(k+i)(t) = \rho(k+i) \ne v$, but $\gamma(k+i+1)(\alpha(k+i)+1) = \gamma(k+i)(t) \ne v$. So we know that $\gamma(k+i+1) \ne \lambda$. But $\gamma(k+i+1)(\alpha(k+i)+2) \in E'$, so again λ must stay at v.
- At time $t = \alpha(k+i) + 3 = \alpha(k+i+1)$, $\gamma(k+i+1)(t) \in Q$ so we also see that λ must stay at v.
- At time $t = \alpha(k+i+1) + 1$, $\gamma(k+i+1)(t) \in E'$ so yet again λ must stay at v, completing the induction.

We can now conclude that $\lambda(t) = v \notin Q$ for the whole time interval $\alpha(k) - 1$ to $\alpha(k+n) + 1 = \alpha(k) + 3n + 1$, inclusive. But this time interval has size $\alpha(k) + 3n + 1 - (\alpha(k) - 2) = 3n + 3$, a period bigger than $3n + 2$ (the endurance of λ), contradicting λ being ℓ-feasible. □

Claim 4. *$\rho : \mathbb{N} \to P$ is cyclic of order n, i.e.,*

$$(\forall k \in \mathbb{N}) \left[\rho(k+n) = \rho(k)\right].$$

Proof. Suppose not, i.e., that there is some $k \in \mathbb{N}$ with $\rho(k+n) \ne \rho(k)$. Now $\rho(k+1), \ldots, \rho(k+n)$ covers all of P by Claim 3, so for some $i \in \mathbb{N}$, $0 < i \le n$, $\rho(k+i) = \rho(k)$. But $i \ne n$ by our assumption. Hence ρ is not injective (one-to-one) on $\{k, k+1, \ldots, k+n-1\}$, since it hits $\rho(k)$ at both k and at $k+i$, so on $\{k, k+1, \ldots, k+n-1\}$, ρ cannot be surjective onto $P = V$ (a set of size n), but this contradicts the conclusion of Claim 3. □

From this we can conclude that $\rho' = \rho \restriction \mathbb{Z}_{n+1}$ is a path through G that is onto V and that satisfies $\rho'(n) = \rho'(0)$, i.e., that ρ' is a Hamiltonian circuit in G and witnesses that G is a Hamiltonian graph.

□

4. HARDNESS

We can now reiterate the main results of [13] about the difficulty of the PBSPCC (Theorem 5.2 of [13]).

Theorem 4.1. *Let $p : \mathbb{N} \to \mathbb{N}$ be an increasing polynomial where $p(n) \geq 3n^2 + 3n$ for $n \geq 2$. Then the following statements are true:*

1. *The PBSPCC of finding minimum size patrol covers is $\mathcal{NP}$-hard.*
2. *The PBSPCC decision problem is $\mathcal{NP}$-hard.*
3. *The p-bounded cyclic PBSPCC decision problem is $\mathcal{NP}$-complete.*

Proof. Note that $\mathcal{T}(\cdot)$ is an $O(n)$ transformation, as N' has only grown by a fixed factor of approximately 3. We will show that all three problems are $\mathcal{NP}$-hard by demonstrating that $\mathcal{T}(\cdot)$ is a transformation from the Hamiltonian graph decision problem which is itself $\mathcal{NP}$-complete (see [6]).

Suppose that $G = (V, E)$ is a graph and we ask whether G has a Hamiltonian circuit. Let $n = |V|$ and construct $\mathcal{T}(G) = N' = (V', E', P, Q)$ and form a patrolled network (N', ℓ), where $\ell = (3n + 2, 1, d)$ and $d(e') = 1$ for all $e' \in E'$. Then:

1. For the PBSPCC we look for the minimum patrol cover size for (N', ℓ) and test whether that minimum is $n + 1$.
2. For the PBSPCC decision problem we ask whether there is a patrol cover of size $n + 1$.
3. For the p-bounded cyclic PBSPCC decision problem we ask whether there is a patrol cover of size $n + 1$ where the cover is cyclic of order at most $p(n)$.

If G has a Hamiltonian circuit, then by Lemma 3.1 we can construct such a patrol cover of size $n+1$, with each element of that cover being cyclic of order $3n(n + 1)$, which is less than $p(n)$, so for each problem there is definitely such a cover. Note that for the more general PBSPCC minimization problem we cannot have a cover of size $\leq n$, otherwise we would have each vessel patrolling patrol regions without any prospect of leaving to replenish. Conversely, if there is a cover of size $n + 1$ of (N', ℓ), then Lemma 3.2 allows us to conclude that G is Hamiltonian. This demonstrates that all three problems are $\mathcal{NP}$-hard because any polynomial time solution to them will allow us to decide the Hamiltonian graph decision problem.

To prove that the p-bounded cyclic PBSPCC decision problem is $\mathcal{NP}$-complete, we just need show that this problem is in $\mathcal{NP}$. It is easy to test whether a set of routes combines to form a patrol cover over a fixed set of times, so we just need to show that a set of routes can have a description whose size is bounded by a polynomial. For (N', ℓ) we let $\text{size}(N') := |V'| + |E'|$. By fixing p before we state the problem, we ensure that we only need to describe the states of at most $p(\text{size}(N')) \times \text{size}(N')$ different cycles – as a cover, if it exists, will be at most a size where each time step and each patrol region gets its own dedicated patrol vessel – and each cycle will need at most $p(\text{size}(N')) \times \text{size}(N')$ pieces of information to describe it (as we only need to describe what it does at each of its $\leq p(\text{size}(N'))$ time points). Therefore, we only need to test an input of size at most a constant multiple of $(p(\text{size}(N')) \times \text{size}(N'))^2$, which is a polynomial. □

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