Australian Government
Department of Defence
Defence Science and
Technology Organisation

Generating Correlated Gamma Sequences for Sea-Clutter Simulation

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Electronic Warfare and Radar Division

Defence Science and Technology Organisation

DSTO–TR–2688

ABSTRACT

This report presents a hybrid method for simulating sequences of correlated Gamma random variables for modelling sea clutter, using a combination of linear and/or non-linear transforms. Depending on the shape parameter, this method minimises the use of non-linear transformations. Mathematically the method is simpler than its counterpart methods which leads to a quicker simulation run time. Two memoryless non-linear transform (MNLT) approaches are also studied with comparative results showing that the hybrid approach is more computationally efficient and slightly more accurate for low shape parameters. The drawback of the proposed method is, however, that it can only handle positive correlations whilst the two MNLT methods are capable of handling both positive and negative correlations.

Published by

DSTO Defence Science and Technology Organisation
PO Box 1500
Edinburgh, South Australia 5111, Australia

Telephone: (08) 7389 5555
Facsimile: (08) 7389 6567

AR No. 015-270
March 2012

APPROVED FOR PUBLIC RELEASE

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Executive Summary

To support the Generic Phased Array Radar Modelling (GPARM) simulation program of the Defence Science and Technology Organisation (DSTO), which in turn supports many Australian defence programs including SEA1448 (ANZAC ASMD), AIR7000 (future maritime patrol and response capability) and AIR5077 (Wedgetail), this report proposes a hybrid method and examines other existing methods for simulating sequences of correlated random variables with a Gamma distribution.

Radar sea-clutter is a dominant undesired signal seriously affecting radar performance over the sea surface. It is now widely accepted that in most scenarios, radar sea-clutter can be modelled as a compound non-Gaussian random process. One of the most popular is the compound K-distribution which consists of two parts: a fast-varying component representing sea-clutter speckle which is modelled as a complex Gaussian with zero mean and unit variance and a slowly-varying component which is Gamma distributed and represents the underlying sea-clutter intensity. These two components are assumed to be mutually independent.

Depending on radar parameters and sea surface conditions, each component of the received sea-clutter may be correlated and appropriate correlation models should be included in the simulation. For K-distributed sea-clutter, methods for simulating both correlated Gaussian and Gamma processes are required. Simulation of the former process is straightforward and can be realised by a linear transform using either spherically invariant random processes (SIRP) or Fourier synthesis. The advantage of the linear transform is that the desired correlation properties are easily maintained. However, simulation of the correlated Gamma distribution is more difficult and may require application of the so-called memoryless non-linear transform (MNLT) to generate the desired correlation.

This report proposes a hybrid method for simulating sequences of correlated Gamma random variables. The approach depends on both the desired shape parameter and in some cases the correlation coefficient. For most distributions with a shape parameter greater than 0.5, the method only requires linear transforms to generate the desired Gamma correlation. However in other cases, such as when the shape parameter is less than 0.5, the method requires the MNLT to achieve the desired correlation. The MLNT technique proposed in this report is different from its counterparts. Compared to the other methods, the proposed one is mathematically simpler making its numerical implementation easier and more computationally efficient. The drawback is, however, that it can only handle positive correlation coefficients.

Two implementations of the MNLT are also examined and evaluated in this report. The first directly implements the numerical integration described by Tough and Ward, while the second is a polynomial auto-correlation method extended from the former by Weinberg and Gunn. Results for the former have a high level of accuracy and speed due to the optimised code. The latter also performs reasonably well except for small shape parameters less than 0.3. Whilst these methods are much more complex then the hybrid method, they are able to handle negative correlation coefficients. A comparison between methods shows that the hybrid method has a reduced simulation run time and provides slightly higher accuracy in generating the desired correlated sequences, particularly for small shape values.

Another contribution in this report is the provision of criteria for realisable correlation functions. It is shown that not every correlation function can be simulated but only those that have a positive semi-definite or positive definite covariance matrix.

This report serves as a detailed reference for computer programmers and research scientists who want to implement correlated Gamma processes to simulate distributions such as the K-distribution.

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Yunhan Dong received his Bachelor and Master degrees in 1980s in China and PhD in 1995 at the UNSW, Australia, all in electrical engineering. He then worked at the UNSW from 1995 to 2000, and Optus Telecommunications Inc. from 2000 to 2002. He joined DSTO as a Senior Research Scientist in 2002. His research interests are primarily in radar signal and image processing and clutter analysis. Dr Dong was a recipient of both the Postdoctoral Research Fellowships and Research Fellowships from the Australian Research Council.

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1 Introduction

The compound K-distribution is often used to model radar sea-clutter. It consists of a fast-varying component representing sea-clutter speckle that is modelled as a Gaussian process with zero mean and unit variance and a slowly-varying component representing fluctuation of the underlying sea-clutter intensity that is Gamma distributed.

Depending on radar parameters and sea surface conditions, the received sea-clutter may be correlated, with each of the two components making contributions to the correlation. Using a pulsed Doppler radar, the received sea-clutter data in a coherent processing interval (CPI) is in general a two-dimensional data-set[1] (time-range or Doppler-range). The temporal correlation refers to the correlation between pulses separated in time, while the spatial correlation refers to the correlation between range bins. The definition and calculation of both the temporal and spatial correlations is given in Appendix A.

Methods for simulating correlated K-distributed sea-clutter are a highly relevant topic for the radar community. Since the fast-varying component is Gaussian distributed, simulating its correlation is relatively easy using the well-established spherically invariant random process (SIRP) [Rangaswamy, Weiner & Ozturk 1993, Antipov 1998]. On the contrary, the simulation of correlated non-Gaussian processes is not straightforward. Methods for simulating these stochastic processes can be classified into two categories. The first uses the SIRP formulation [Rangaswamy, Weiner & Ozturk 1993] and requires the processes to be linearly transformable from the Gaussian process. To implement, a linear transform relationship is first established between the desired and Gaussian distributions and then the correlation of the desired distribution is mapped onto the correlation of the Gaussian process. The correlated Gaussian process is then realised through SIRP before being transformed back to the desired distribution. Since the linear transform maintains the correlation properties, SIRP can achieve a high level of accuracy. However, only a few random stochastic processes can be linearly connected to the Gaussian process.

The generation of correlated Gamma random fields via SIRP theory is examined in [Conte et al. 1991, Armstrong & Griffiths 1991]. In these papers, the Gamma random process is expressed as a sum of Gaussian processes when the Gamma shape parameter ν has special values. The method of [Conte et al. 1991] shows that the Gamma distribution, in the range of $0 < \nu \leq 2$, closely resembles a Gaussian process and hence the correlated Gamma with a shape parameter in that range can be produced via SIRP. On the other hand, the method of [Armstrong & Griffiths 1991] only considers a special correlation case where the correlation follows a geometric progression (i.e., the correlation coefficients form a geometric sequence) for ν, having values of $\nu = 0.5n$ where n is a natural number.

The second category of methods for simulating correlated random stochastic processes is the memoryless non-linear transformation (MNLT). This method is examined extensively in [Tough & Ward 1999] for simulating different correlated non-Gaussian variates including Gamma. The difficulty of the MNLT lies in the non-linear mapping of the auto-correlation of the output (the desired non-Gaussian process) to the auto-correlation of the input (the Gaussian process).

This report proposes a hybrid method for simulating correlated Gamma random variables. The approach depends on both the desired shape parameter and in some cases the correlation coefficient. For most distributions with a shape parameter greater than 0.5, the method only requires

[1] It can be three-dimensional if radar has a multi-channel receiver.

linear transforms of SIRP to generate the desired Gamma correlation. However in other cases, such as when the shape parameter is less than 0.5, the method requires the MNLT to achieve the desired correlation.

Two implementations of the MNLT are also examined and evaluated in this report. The first directly implements the approximation of the non-linear mapping described in [Tough & Ward 1999]. This method is referred to as the MNLT with numerical integration. The second method is described in [Weinberg & Gunn 2011*a*] and extends this method with a polynomial approximation to allow near optimal control between the input and output processes. This method is referred to as the MNLT with polynomial approximation.

The report is organised as follows. In Section 2, the proposed hybrid method for simulating correlated Gamma random variables is described. The method extends those of [Conte et al. 1991, Armstrong & Griffiths 1991] to enable an arbitrary shape parameter to be obtained through SIRP for most cases. Details of the two MNLT methods are then briefly described in Section 3 with evaluation and comparison of the three methods presented in Section 4.

2 The Hybrid Method

A random variable Z is said to have a Gamma distribution if it has a probability distribution function (PDF) of,

$$p_Z(z) = \begin{cases} \dfrac{1}{\gamma(\nu)\theta^\nu} z^{v-1} \exp\left(-\dfrac{z}{\theta}\right), & z > 0, \\ 0 & z < 0, \end{cases} \tag{1}$$

where $\gamma(\nu) = (\nu - 1)!$ is the Gamma function. The above Gamma distribution is commonly denoted by $Z \sim \Gamma(\nu, \theta)$, where ν is the shape parameter and θ the scale parameter. It is known that the Gamma distribution has the following property:

Lemma 1: If $Z_k \sim \Gamma(\nu_k, \theta)$ for $k = 1, \ldots K$ (i.e., all have the same scale parameter, but may have different shape parameters), and all Z_k are mutually independent, then $\sum_{k=1}^{K} Z_k \sim \Gamma(\nu, \theta)$, where $\nu = \sum_{k=1}^{K} \nu_k$.

Lemma 1 only requires that Z_k are mutually independent and not whether sequences of Z_k are auto-correlated. Consider samples of the random process Z_k taken at intervals $t = n\Delta_t$ where $1/\Delta_t$ is the sampling rate. A discrete sequence is defined by $z_k[n] \sim \Gamma(\nu, \theta)$ where n is a natural number. The following two properties reveal the relationship between Gamma and Gaussian processes and thus SIRP can be established from Gaussian to Gamma.

Property 1: If X is a real Gaussian variable, $X \sim N(0, \theta/2)$, then its intensity $Z = |X|^2$ has a Gamma distribution of $Z \sim \Gamma(0.5, \theta)$.

Property 2: If X is a complex Gaussian variable, $X \sim CN(0, \theta)$, then its intensity Z has a Gamma distribution of $Z \sim \Gamma(1, \theta)$.

Strictly speaking, sea-clutter in general is not a wide-sense stationary (WSS) process. However, the correlation of the Gamma can be assumed stationary. The goal of the hybrid method is to generate a sequence $z[n]$ that has a $\Gamma(\nu, \theta)$ distribution and a correlation coefficient ρ_k given by,

$$\rho_k = \frac{E\{z[n]z[n+k]\} - \bar{z}^2}{\text{var}(z)} \quad k = 0, 1, \ldots \text{ and } \rho_{-k} = \rho_k. \tag{2}$$

where var(z) is the variance of z. The relationship between the covariance $E\{(z[n] - \bar{z})(z[n+k] - \bar{z})\}$ and the correlation coefficient ρ_k is governed by (1). Since $z[n]$ is Gamma distributed, as $Z \sim \Gamma(v, \theta)$, $\bar{z} = v\theta$, $\overline{z^2} = v\,\theta^2 + v^2\theta^2$ and var$(z) = v\theta^2$, then $\rho_0 = 1$ and $|\rho_k| \leq 1$ for $k > 0$. Note that if the correlated Gamma is formed by linear correlated Gaussian processes, the lowest bound of ρ_k achievable is inherently 0 which means $E\{z[n]z[n+k]\} \geq E^2\{z[n]\}$. This will become clear in the next subsection[2].

The approach of the hybrid method utilises these properties to efficiently produce the correlated random sequence. The exact approach depends on both the desired shape parameter and in some cases the correlation coefficient. For most distributions with a shape parameter greater than 0.5, the method only requires linear transforms of SIRP or Fourier synthesis to generate the desired correlation. However in other cases, such as when the shape parameter is less than 0.5, the method requires the MNLT to achieve the desired correlation. The following three cases describe this in more detail.

[2]However, if correlated Gamma is formed by the MNLT methods discussed in Section 3, $E\{z[n]z[n+k]\} < E^2\{z[n]\}$ and $\rho_k < 0$ is achievable.

2.1 Case I – Shape parameter to be an integer

The first case considers an integer shape parameter. With this restriction, the method of [Armstrong & Griffiths 1991] first introduced the transform between a correlated Gaussian and correlated Gamma. However, their method only considered the case of geometrical correlation. In this section, the linear method is extended to include any realisable correlation function but limited to $\rho_k \geq 0$. The criteria for realisable correlations are given in Subsection 4.1.

Since sequences with different scale parameters can be obtained by re-scaling[3], $\theta \equiv 1$ in the following discussions. For the special case of $\nu = 1$, $Z \sim \Gamma(1,1)$ and the sequence of $z[n]$ is found by $z[n] = |x[n]|^2$, where $x[n]$ is a sequence of the Gaussian process and $X \sim CN(0,1)$. Therefore,

$$\langle z[n]z[n+k]\rangle = \langle x[n]x^*[n]x[n+k]x^*[n+k]\rangle \tag{3}$$

where the superscript * denotes complex conjugate. Since $x[n]$ is a complex Gaussian, the expectation of the right side of (3) can be found by employing Isserlis's theorem [Michalowicz et al. 2009], giving,

$$\langle x[n]x^*[n]x[n+k]x^*[n+k]\rangle = \left\langle |x|^2 \right\rangle^2 + |\langle x[n]x^*[n+k]\rangle|^2 . \tag{4}$$

Because $\left\langle |x|^2 \right\rangle^2 = \bar{z}^2$ and $|\langle x[n]x^*[n+k]\rangle|^2 \geq 0$, this results in $\langle z[n]z[n+k]\rangle \geq \bar{z}^2$ if the realisation of $z[n]$ is through $z[n] = |x[n]|^2$. Therefore, regardless of the correlation of the Gaussian $\langle x[n]x^*[n+k]\rangle$, $\rho_k \geq 0$ is the lowest bound of correlation coefficient achievable for $Z \sim \Gamma(1,1)$ if its realisation is formed by taking the intensity of correlated Gaussian $X \sim CN(0,1)$. It can be seen that by taking the intensity of a Gaussian variable to form a Gamma variable inherently limits the low bound of the correlation coefficient ρ. The correlation of Gaussian is,

$$\langle x[n]x^*[n+k]\rangle = \pm\left(\langle z[n]z[n+k]\rangle - \bar{z}^2\right)^{1/2} = \pm\left(\rho_k \text{var}(z)\right)^{1/2}, \qquad k = 0, 1, \ldots . \tag{5}$$

Because $\rho_k \text{var}(z) \geq 0$, its square root is a non-negative real number. If $\mathbf{x} = [x[1], x[2], \ldots]^T$ is a column vector, where the superscript T denotes transpose, then

$$E\left\{\mathbf{x}\mathbf{x}^H\right\} = \mathbf{M}_x, \tag{6}$$

where $\mathbf{M}_x$ is the covariance matrix of $\mathbf{x}$ and the superscript H denotes the Hermitian transpose. According to the assumptions of stationarity and symmetry, $\mathbf{M}_x$ has a Toeplitz structure, with elements given by (5), i.e., $\mathbf{M}_x(n,k) = \left(\rho_{|n-k|}\right)^{1/2}$ (since var$(z) = 1$ for $Z \sim \Gamma(1,1)$). According to SIRP, let,

$$\mathbf{x} = \mathbf{M}_x^{1/2}\mathbf{u} \tag{7}$$

where $U \sim CN(0,1)$ is an uncorrelated Gaussian process. After obtaining the correlated $\mathbf{x}$ and applying the square-law to each element, the sequence (or the column vector) of $\mathbf{z} = \left[|x[1]|^2, |x[2]|^2, \ldots\right]^T$ then has the desired correlation of,

$$E\left\{(\mathbf{z} - \bar{z})(\mathbf{z} - \bar{z})^T\right\} = \text{var}(z)\,\mathbf{M}_z \tag{8}$$

[3] i.e., if $Z \sim \Gamma(\nu, 1)$, then $\theta Z \sim \Gamma(\nu, \theta)$.

where $\mathbf{M}_z = \text{Toeplitz}(\boldsymbol{\rho})$, i.e., $\mathbf{M}_z(n,k) = \rho_{|n-k|}$, which is the same as the desired correlation in (2).

The above correlated Gaussian sequences can also be obtained through Fourier synthesis [Ward, Tough & Watts 2006, Section 5.4]. The application of a filter h to a white Gaussian noise process $u(t)$ gives,

$$x(t) = \int_{-\infty}^{\infty} h(t')u(t')dt'. \tag{9}$$

In the frequency domain, the above equation is written as,

$$X(f) = H(f)U(f) \tag{10}$$

where $U(f)$ is still a Gaussian process. On the other hand, according to the Wiener-Khinchin theorem [Haykin 2007, Chapter 2], the power spectrum density (PSD) of the correlated signal $x(t)$ is,

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_G(t')e^{-j2\pi ft'}dt' \tag{11}$$

where $R_G(t)$ is the auto-correlation function of $x(t)$. Selecting $H(f) = \sqrt{S_{xx}(f)}$, the Fourier inversion of $X(f) = H(f)U(f)$ will then result in a correlated Gaussian process $x(t)$ with the desired correlation $R_G(t)$.

The advantage of the Fourier synthesis against SIRP is that the Fourier transform in the numerical simulation can be realised through the fast Fourier transform (FFT) and hence significantly improves the computational efficiency. In this report, the hybrid method will use the Fourier synthesis instead of SIRP to generate correlated Gaussian sequences.

If $\nu = m > 1$, where m is an integer, Lemma 1 can be utilised to repeatedly generate $\mathbf{x}_k$, $k = 1, \ldots, m$. The result $z[n] = \sum_{k=1}^{m} |x_k[n]|^2$ will then have a $\Gamma(m, 1)$ distribution with the desired correlation.

2.2 Case II – Shape parameter to be an integer plus 0.5

If the shape parameter has a value of $\nu = m + 0.5$, where $m \geq 0$ is an integer, the sequence $\mathbf{x}_k$ is repeatedly generated for $k = 1, \ldots, m + 1$. The result $z[n] = \sum_{k=1}^{m} |x_k[n]|^2 + \text{Re}^2\left(x_{m+1}[n]\right)$ will then have a $\Gamma(m + 0.5, 1)$ distribution with the desired correlation (according to Lemma 1 and Properties 1 and 2).

2.3 Case III – Shape parameter to be an arbitrary number

If the shape parameter takes an arbitrary value, $\nu = \nu_0 + \Delta\nu$, where $\nu_0 = m + 0.5\beta$, $m \geq 0$ is an integer, $\beta = 0$ or 1 and $0 < \Delta\nu < 0.5$. Two sequences, $y[n]$ and $\Delta y[n]$ can then be simulated with respective shape values ν_0 and $\Delta\nu$ so their summation produces a sequence with the desired shape and correlation.

The goal of the hybrid method is to use linear transforms whenever possible to generate the sequences of random variables. The criteria for whether the linear method will work is determined by $0 \leq \eta_k \leq 1$ where η_k and ρ_k are related by,

$$\eta_k = \begin{cases} 1 & k = 0 \\ \rho_k \nu / \nu_0 & k \neq 0 \end{cases}, \tag{12}$$

with the correlation coefficient η_k of the sequence $y[n]$ given by,

$$\eta_k = \frac{\langle y[n]y[n+k]\rangle - \bar{y}^2}{\text{var}(y)} \qquad k = 0, 1, \ldots. \tag{13}$$

If this relationship is valid, the hybrid method uses the Fourier synthesis method to simulate the desired sequence. On the other hand, it is possible that this relationship will not be satisfied and a MNLT technique must be employed. Each case can be summarised by three steps:

Linear Case

Step 1: If the relationship in (12) is satisfied, generate a correlated Gamma sequence of $Y \sim \Gamma(\nu_0, 1)$ which has a correlation coefficient of η_k.

Step 2: Generate an uncorrelated Gamma sequence of $\Delta Y \sim \Gamma(\Delta\nu, 1)$. Since the sequence is uncorrelated, the generation of such a sequence is easy to implement by simply using either the existing function in MATLAB, `gamrnd`, or other MNLT methods [Ahrens & Dieter 1974].

Step 3: Combine the correlated sequence with $\Gamma(\nu_0, 1)$ distribution and the uncorrelated sequence with $\Gamma(\Delta\nu, 1)$ distribution to produce $z[n] = y[n] + \Delta y[n]$ with distribution $\Gamma(\nu, 1)$.

To see how the aggregate correlation of $\langle (y[n] + \Delta y[n])\,(y[n+k] + \Delta y[n+k])\rangle$ is equal to the correlation of $\langle z[n]z[n+k]\rangle$, notice that the sequence $y[n]$ is correlated whereas the sequence of $\Delta y[n]$ is uncorrelated and the two sequences are mutually independent. Equation (13) then becomes,

$$\eta_k = \frac{\rho_k \text{var}(z) + \bar{z}^2 - 2\Delta\bar{y}\bar{y} - \bar{y}^2 - \langle \Delta y[n]\Delta y[n+k]\rangle}{\text{var}(y)}, \tag{14}$$

where $\text{var}(z) = \nu\theta^2$, $\bar{z} = \nu\theta$, $\Delta\bar{y} = \Delta\nu\theta$, $\bar{y} = \nu_0\theta$, $\text{var}(y) = \nu_0\theta^2$ and

$$\langle \Delta y[n]\Delta y[n+k]\rangle = \begin{cases} \Delta\nu\theta^2 + \Delta\nu^2\theta^2 & k = 0 \\ \Delta\nu^2\theta^2 & k \neq 0 \end{cases}. \tag{15}$$

Inserting these values in (14) results in (12).

MNLT Case

Step 1: If the relationship in (12) is not satisfied, generate a correlated Gamma sequence of $Y \sim \Gamma(\nu_0, 1)$ which has a correlation coefficient of ρ_k.

Step 2: Generate a correlated Gamma sequence of $\Delta Y \sim \Gamma(\Delta\nu, 1)$ using the MNLT method described in Section 2.4.

Step 3: Combine the correlated sequence with $\Gamma(\nu_0, 1)$ distribution and the correlated sequence with $\Gamma(\Delta\nu, 1)$ distribution to produce $z[n] = y[n] + \Delta y[n]$ with distribution $\Gamma(\nu, 1)$.

2.4 A MNLT technique for shape parameter $0 < \Delta\nu < 0.5$

As mentioned previously, not every case of correlated Gamma can be tackled by the Linear method and the MNLT must be used instead. This subsection introduces a MNLT technique specially designed for tackling the case of $0 < \Delta\nu < 0.5$.

The MNLT is a one-to-one non-linear mapping from a known random process (not necessarily Gaussian) to another random process with a desired density function. Supposing a random process $y[n]$ has a known PDF $p_Y(y)$. The MNLT uses this PDF to generate a random process $\Delta y[n]$ with a desired PDF $p_{\Delta Y}(\Delta y)$. This may be realised by the mapping of,

$$\int_{\Delta y[n]}^{\infty} p_{\Delta Y}(\Delta y')d\Delta y' = \int_{y[n]}^{\infty} p_Y(y')dy' \quad n = 1, 2, \ldots. \tag{16}$$

In principle, the MNLT method can be used to generate any non-Gaussian process. In the following discussion however, the method is limited by using the correlated $\Gamma(1, 1)$ distribution to generate a $\Gamma(\Delta\nu, 1)$ distribution with the same correlation.

Using the linear transform given in the previous section, a sequence $y[n]$ is first generated, (note $Y \sim \Gamma(1, 1)$) with a desired correlation coefficient of ρ_k. The one-to-one non-linear mapping of (16) is then used to produce the corresponding $\Delta y[n]$, with a PDF of $\Gamma(\Delta\nu, 1)$. This is done by equating their complementary distribution functions (CDF) described in (16) giving,

$$\frac{1}{\gamma(\Delta\nu)}\gamma_I\left(\Delta\nu, \Delta y[n]\right) = \exp\left(-y[n]\right), \quad n = 1, 2, \ldots \tag{17}$$

where $\gamma_I(a, b)$ is the upper incomplete Gamma function, defined as,

$$\gamma_I(a, b) = \int_b^{\infty} t^{a-1}e^{-t}dt. \tag{18}$$

For a Gamma distribution with an integer shape parameter, or an integer plus 0.5, the correlation given by (5) is independent of the shape parameter. In another words, the linear transform does not change the correlation. According to our simulation however, it was found that the MNLT of (17) does slightly alter the correlation. As a consequence, while the correlation of $y[n]$ has the desired values, after the non-linear transform, the correlation of $\Delta y[n]$ will have slightly different values. A rigid mathematical trace of how the correlation is being changed through the above MNLT is difficult and hence an empirical formula is introduced to maintain the desired correlation.

Consider the example of generating a random process $\Delta y[n]$ using the MNLT of (17) with the mapping seeds of $y[n]$. The former has a $\Gamma(\Delta\nu, 1)$ distribution with a correlation coefficient of ρ_k, and the latter has a $\Gamma(1, 1)$ distribution with a correlation coefficient of ψ_k. The empirical relationship between ρ_k and ψ_k is given by,

$$\psi_k = \rho_k^{(\Delta\nu+0.5)^{0.3}}, \quad k = 0, 1, \ldots \quad \text{for } 0 < \Delta\nu < 0.5 \tag{19}$$

and is a function of both ρ_k and $\Delta\nu$. The transformation from ρ_k to ψ_k is minor as shown in Figure 1. When $\Delta\nu$ approaches 0.5, ψ and ρ are identical, since the transform from $\Gamma(1, 1)$ to $\Gamma(0.5, 1)$ is linear.

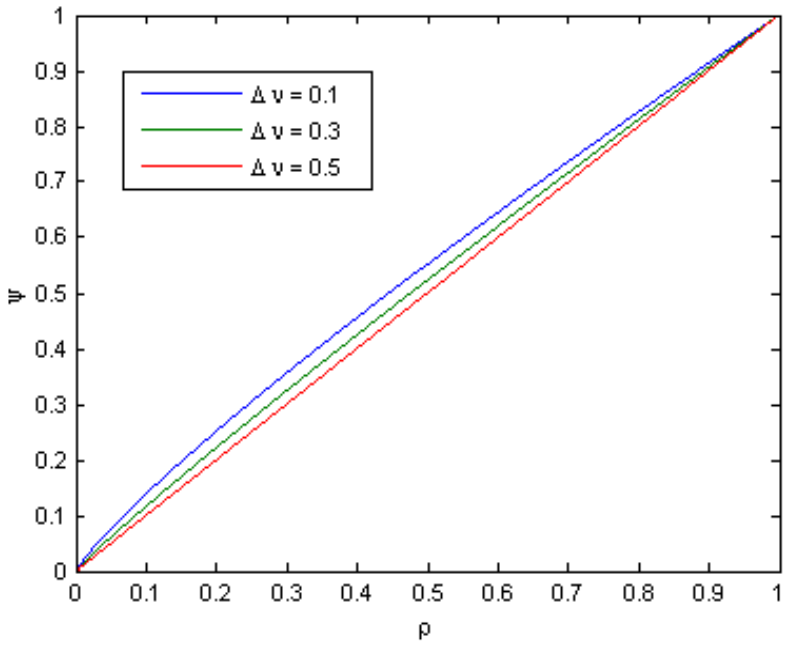


Figure 1: *The relationship between ρ and ψ*

2.5 Overview of the Hybrid Method

In summary, the proposed hybrid method for simulating correlated Gamma sequences primarily depends on the shape parameter and may depend on the correlation coefficient. It is summarised by the following three cases:

Case I If $\nu =$ integer, generate the desired correlated Gamma using the linear Fourier method;

Case II If $\nu =$ integer $+ 0.5$, generate the desired correlated Gamma using the linear method;

Case III If ν is an arbitrary number, let $\nu = \nu_0 + \Delta\nu$, where $0 < \Delta\nu < 0.5$ and ν_0 is an integer or an integer plus 0.5.

- If $0 \leq \rho_k\nu/\nu_0 \leq 1$ is satisfied, generate a correlated Gamma of shape ν_0 with the correlation of (13) following cases I or II. Generate a second uncorrelated Gamma sequence with shape $\Delta\nu$ and sum the two sequences.
- If the check fails, generate a correlated Gamma of shape ν_0 with the correlation of ρ_k following cases I or II. Generate a second correlated sequence using the MNLT technique described in Subsection 2.4 to generate a correlated Gamma with shape $\Delta\nu$. Sum the two sequences together.

Therefore, it can be seen that the generation of correlated Gamma only requires linear transform of Gaussian process for the majority of cases to efficiently produce the correlated random sequences.

3 Simulation with Memoryless Non-Linear Transform

The concept of MNLT described in (16) is a non-linear mapping from a known random stochastic process (input) to an unknown one (output) by equating their CDF's. If Ξ denotes the desired random process, then its relation to the Gaussian process X (its intensity is an Exponential distribution), is given by,

$$\int_{\xi}^{\infty} p_{\Xi}(\xi')d\xi' = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} \exp\left(-\frac{x'^2}{2}\right) dx' = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right), \tag{20}$$

where p_{Ξ} is the PDF of Ξ. The reason to often choose a Gaussian process as the input, is that its correlation can be readily realised through SIRP or Fourier synthesis. The difficulty of MNLT for correlated processes is that the mapping of the correlation is also non-linear. In particular, the realisation of a correlated Gamma needs to first transfer the correlation of the Gamma into the correlation of a Gaussian. The technique for manipulating the transform of the output correlation back to the input can result in different approaches for the MNLT. In this section, two approaches are presented. The first is based on using numerical integration and the other employs inverted polynomials to map the correlation of the output to the correlation of the input.

3.1 MNLT Approach I – Numerical Integration

This approach was first described in [Tough & Ward 1999] and expanded in more detail in Chapter 5 of [Ward, Tough & Watts 2006]. The first step of the approach is to use Fourier synthesis to produce correlated sequences of random numbers with a Gaussian distribution. The real and imaginary parts of the inverse Fourier transform provide separate realisations of the complex Gaussian process. The MNLT which maps the input Gaussian random variables to the output non-Gaussian random variables is via (20) and can be written as,

$$\xi(x) = Q_{\text{dist}}\left(\text{erfc}\left(\frac{x}{\sqrt{2}}\right)/2\right), \tag{21}$$

where the function Q_{dist} is the complementary quantile function for the output distribution and $\text{erfc}(\cdot)$ is the complementary error function. Rapid evaluation of this function is essential for practical implementation of the method. For the Gamma distribution, Q_{dist} is the inverse incomplete Gamma function. An efficient algorithm for this function was developed and implemented in Fortran by DiDonato & Morris [1986]. The Fortran code and a C translation, is available on the web as part of the Cumulative Distribution Function Library (CDFLIB) [Venier & Serachitopol 2003]. The C code has been compiled as a `Mex` file for MATLAB by Davidson [2011] as part of his Radar Toolbox. Two special cases are implemented with MATLAB intrinsic functions: the Exponential distribution ($\nu = 1$), where $Q_{\text{dist}}(u) = -\log(u)$, and the Chi-square distribution ($\nu = 1/2$), where $Q_{\text{dist}}(u) = \text{erfcinv}^2(u)$ and $\text{erfcinv}(\cdot)$ is the inverse complementary error function.

The desired correlation function for the output non-Gaussian process must be mapped to the correlation function for the input Gaussian process. The construction of this mapping is described in Section 5.6 of [Ward, Tough & Watts 2006]. It requires the evaluation of integrals in the form of [Tough & Ward 1999],

$$\langle \xi(0)\xi(t)\rangle = \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{R_G(t)^p}{2^p p!}\int_{-\infty}^{\infty}\exp\left(-x'^2\right)H_p(x')Q_{\text{dist}}\left(\frac{\text{erfc}(x')}{2}\right)dx', \tag{22}$$

where $R_G(t) = \langle x(0)x(t)\rangle / \langle x^2\rangle$ is the correlation coefficient of $x(t)$ (note x is real and $X \sim N(0,1)$) and $H_p(\cdot)$ is the Hermite polynomial of order p. The Hermite polynomial coefficients are obtained with the function `HermitePoly` written by David Terr, found on MATLAB Central [Terr 2004]. The integrals are evaluated with 12 point Gauss-Hermite integration, using the function `GaussHermite`, which is a MATLAB translation of the Fortran function `hermite.f90` by Miller [2010].

The advantage of this MNLT is that Gamma sequences with $\rho(t) < 0$ can be simulated, though the correlation is still be bounded to the condition given in Subsection 4.1. According to Chapter 5 of [Ward, Tough & Watts 2006], for the case of $\nu = 1$, (22) reduces to,

$$\begin{aligned}\langle \xi(0)\xi(t)\rangle \approx \quad & 1 + 0.816R_G(t) + 0.177R_G(t)^2 + 0.0067R_G(t)^3 \\ & +0.00013R_G(t)^4 + 0.000017R_G(t)^5.\end{aligned} \tag{23}$$

This means that a Gamma process with a correlation $\rho(t) < 0$ can be simulated by choosing a proper $R_G(t) < 0$.

3.2 MNLT Approach II - Polynomial Approximation

When the MNLT converts a correlated Gaussian process into a new process with desired marginal distributions, the new process is also correlated, but not through the same auto-covariance function as the Gaussian process. It is in fact related to the Gaussians auto-covariance by a non-linear mapping. In a practical situation, a process with a given auto-covariance function would be specified. It is shown that by using an appropriate inversion method, the correlation of the Gaussian can be obtained with the desired correlation for the new process [Weinberg & Gunn 2011*a*].

As mentioned, the MNLT method is an extension of the CDF inversion method with which a single random variable can be generated. For a given continuous random variable, inverting its CDF, and evaluating it on uniformly distributed random numbers between 0 and 1, will generate sequences distributed from the original random variable. This can be used to very easily produce independent samples. To extend this to the generation of correlated random samples, it is required that if X_1 and X_2 are two random variables, with distribution functions F_{X_1} and F_{X_2} respectively, then the random variable

$$F_{X_2}^{-1}(F_{X_1}(X_1)) \sim X_2. \tag{24}$$

The validity of this result is discussed in [Weinberg & Gunn 2011*a*]. Equation (24) is the key to generating correlated realisations of a random process. This can be seen by defining a function $\xi(x) = F_{X_2}^{-1}(F_{X_1}(x))$ and observing that if x is a realisation of random variable X_1, then $\xi(x)$ is a realisation of random variable X_2.

Consider a wide sense stationary stochastic process $\zeta(t)$ evolving over time t, with correlation function $R_\zeta(\tau) = \mathbf{E}(\zeta(0)\zeta(\tau))$, and each $\zeta(t) \sim X_1$. This can be transformed to produce a new stochastic process $\theta(t) = \xi(\zeta(t))$, which will be distributed according to X_2 pointwise. However, this new process will have a correlation function $R_\theta(\tau) = \kappa(R_\zeta(\tau))$, where $\kappa(\cdot)$ is a non-linear function.

The advantage of this is that any random variable X_2 can be generated from any X_1. If X_1 is used to generate a correlated Gaussian sequence which is then transformed into a sequence of random variables with the desired marginal statistical distribution using the function ξ. The

resulting sequence will then consist of dependent realisations. Due to the non-linear nature of ξ, the output sequence will not necessarily have the same correlation properties as the Gaussian input process. Consequently, the relationship between input and output auto-covariance functions is examined in detail in [Tough & Ward 1999]. In particular, if the Gaussian process auto-covariance is denoted R_G, and the desired correlated process has auto-covariance R_{out}, then (22) can be written as,

$$R_{\text{out}}(t) = \sum_{p=0}^{\infty} \frac{(R_G(t))^p}{2^p p!} a_p^2, \tag{25}$$

with the coefficients a_p given by

$$a_p = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_p(x) \xi(\sqrt{2}x) dx. \tag{26}$$

The relationship (25) would be more useful if it could be inverted, to specify the Gaussian auto-covariance necessary to generate a desired auto-covariance in the output process. Some simple examples where this relationship can be inverted are considered in [Tough & Ward 1999].

This issue is addressed in detail in the report [Weinberg & Gunn 2011a]. In particular, a polynomial approximation to (25) is derived, which is inverted and scaled, and consequently used to determine an appropriate Gaussian input process. As an example, four terms were manually calculated, resulting in a polynomial approximation of,

$$\begin{aligned} R_G(p) \approx \quad & \tfrac{1}{b_1} \left(R_{\text{out}}(p) - b_0\right) \\ & - \tfrac{b_2}{b_1^3} \left(R_{\text{out}}(p) - b_0\right)^2 \\ & - \left(\tfrac{b_3}{b_1^4} - \tfrac{2b_2^2}{b_1^5}\right) \left(R_{\text{out}}(p) - b_0\right)^3 \\ & - \left(\tfrac{b_4}{b_1^5} - \tfrac{5b_2 b_3}{b_1^6} + \tfrac{5b_2^3}{b_1^7}\right) \left(R_{\text{out}}(p) - b_0\right)^4, \end{aligned} \tag{27}$$

where $b_p = a_p^2/(2^p p!)$.

This has been found to work well in a number of cases considered. Furthermore, the validity of this polynomial approximation has been examined in [Weinberg & Gunn 2011b]. In terms of practical simulation of desired correlated processes, it has been reported that the success of this approach can vary greatly. According to our simulation, it has been found that the approach produces poor results for the correlated Gamma with small shape parameters, $\nu < 0.3$. This may be due to the size of the polynomial used for the approximation. Further investigation is required for this case.

4 Examples and Comparisons

A number of different methods have been described for simulating correlated Gamma random variables. Correlation, however, has its own physical and mathematical constraints which have not been discussed. Section 4.1 presents two constraints to ensure that a given correlation function is mathematically (and possibly physically) realisable. Two examples of applying the hybrid method are then presented in Section 4.2. A performance comparison of the methods is then given in Section 4.3. Finally, a description of how the hybrid method can be used to simulate K-distributed data is given in Section 4.4.

4.1 Realisable Correlation Functions

This section presents two constraints which are required to ensure that a correlation functions is realisable.

Constraint 1: As shown by (2) the correlation coefficient of a correlated Gamma sequence $z[n]$ satisfies $\rho_0 = 1$ and $|\rho_k| \leq 1$ for $k > 0$.

Constraint 2: The correlation matrix $\mathbf{M}_z$ given by (8) is positive semi-definite. The proof is given below.

Under the stationary and symmetric assumptions, the covariance matrix of Gamma distributed random vector $\mathbf{z} = [z[1], \ldots, z[N]]^T$ is given by,

$$\mathbf{\Sigma} = E\left\{(\mathbf{z} - \bar{z})\,(\mathbf{z} - \bar{z})^H\right\} = \text{var}(z)\mathbf{M}_z. \tag{28}$$

For an arbitrary constant (non-random) vector $\mathbf{q} = [q_1, \ldots, q_N]^T$, the quadratic form is,

$$\mathbf{q}^H\mathbf{\Sigma}\mathbf{q} = E\left\{\left(\mathbf{q}^H(\mathbf{z} - \bar{z})\right)\left((\mathbf{z} - \bar{z})^H\mathbf{q}\right)\right\} = E\left\{\left|\mathbf{q}^H(\mathbf{z} - \bar{z})\right|^2\right\} \geq 0. \tag{29}$$

Therefore the covariance matrix $\mathbf{\Sigma}$, or equivalently the correlation matrix $\mathbf{M}_z$ (because of $\text{var}(z) > 0$), is positive semi-definite.

These two constraints are the characteristics of Gamma distributed sequences (wide-sense stationary sequences) and independent of simulation methods. If the hybrid method is used to generate correlated gamma sequences, the aforementioned additional condition of $\rho_k \geq 0$ is also required.

The validity of the correlation using the above conditions must therefore be checked when simulating a correlated Gamma process with a given set of correlation coefficients. Any correlation not satisfying the above constraints cannot be realised accurately by the above methods. For example, consider the correlation coefficient,

$$\rho_k = |\cos(0.1\pi k)|\, e^{-k/10}, \quad |k| = 0, 1, \ldots. \tag{30}$$

A simple check indicates that $\mathbf{M}_z$ is not positive semi-definite and hence the desired correlation cannot be precisely generated by any of the above simulation methods. However, for a correlation coefficient of,

$$\rho_k = \cos(0.125\pi k)e^{-k/10}, \quad |k| = 0, 1, \ldots, \tag{31}$$

the corresponding $\mathbf{M}_z$ is positive definite and hence the desired correlated correlation can be simulated by the MNLT methods. However, due to the condition of $\rho_k > 0$ not being satisfied, the proposed hybrid method is unable to produce the desired correlation.

4.2 Examples with the Hybrid Method

To demonstrate the hybrid method, two numerical examples are now presented. The first has a correlation coefficient of,

$$\rho_k = [0.7 + 0.3\cos(0.12\pi\, k)]\; e^{-k/12}, \qquad k = 0, 1, 2, \ldots. \tag{32}$$

The desired shape parameter is $\nu = 3.7$ which gives $\nu_0 = 3.5$ and $\Delta\nu = 0.2$. A simple check shows the condition of $\eta_k = \rho_k \nu/\nu_0 < 1$ is satisfied. Therefore, the linear transform was used to generate a correlated dataset that has a $\Gamma(3.5, 1)$ distribution with a correlation coefficient of η_k. The second uncorrelated dataset having a $\Gamma(0.2, 1)$ distribution was simply produced by calling the MATLAB function `gamrnd`. The combination of the two will then give a $\Gamma(3.7, 1)$ distribution with the desired correlation coefficient of ρ_k. Even though typical radar systems may only collect a few hundred range bins at a time, a sequence containing 10^6 samples was generated to ensure there are sufficient samples for confirming the actual correlation. The comparison between the desired and simulated correlations, probability and cumulative densities are shown in Figure 2. It can be seen that the desired correlation of the Gamma has been precisely achieved. The second

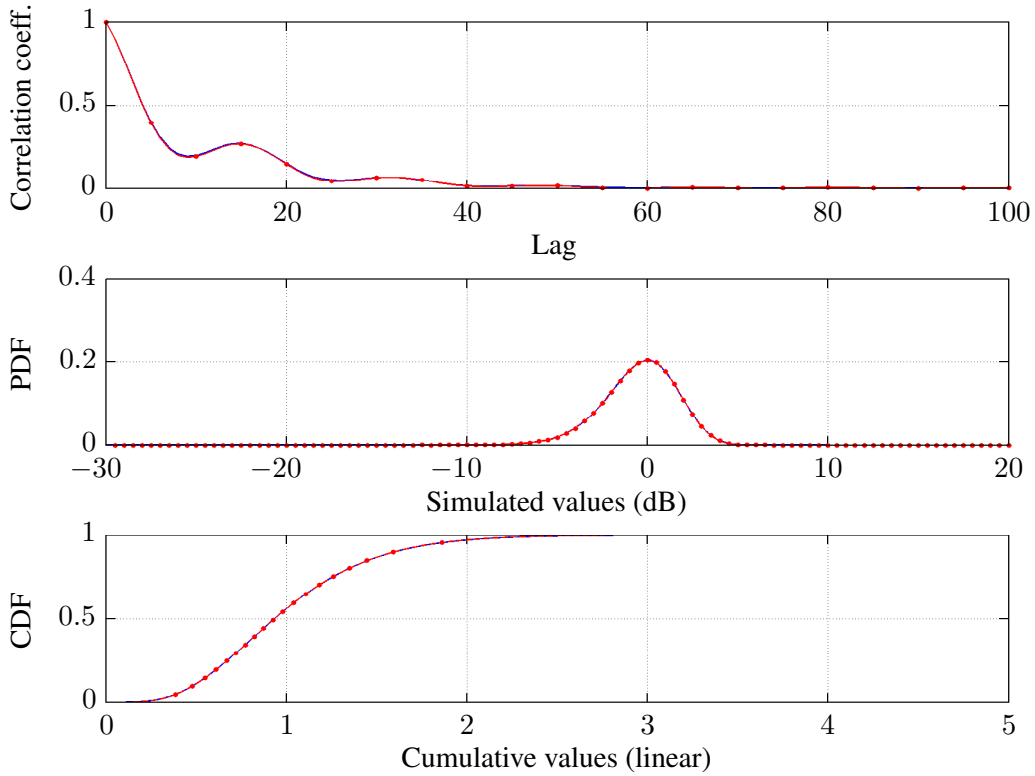


Figure 2: *Comparison between the desired and simulated correlation coefficients for Example 1: (—) exact, (- - -) simulated: (top) one-sided correlation coefficients; (middle) PDF's of the Gamma and (bottom) CDF's of the Gamma.*

example has a correlation coefficient of,

$$\rho_k = e^{-k/10}, \quad k = 0, 1, 2, \ldots. \tag{33}$$

In this case, the desired shape parameter is $\nu = 0.7$ which gives $\nu_0 = 0.5$ and $\Delta\nu = 0.2$. A simple check shows the condition of $\eta_k = \rho_k\nu/\nu_0 < 1$ is not satisfied. Therefore, the linear transform is used to generate a correlated dataset that has a $\Gamma(0.5, 1)$ distribution and the MNLT method described in Subsection 2.4 is used to generate a correlated dataset that has a $\Gamma(0.2, 1)$ distribution. Both of them have a correlation coefficient of ρ_k. The combination gives a dataset that has a $\Gamma(0.7, 1)$ distribution and the desired correlation. The comparison between the desired and simulated correlation coefficients, probability and cumulative densities are shown in Figure 3. The results again show that the desired correlation of the Gamma has been precisely achieved.

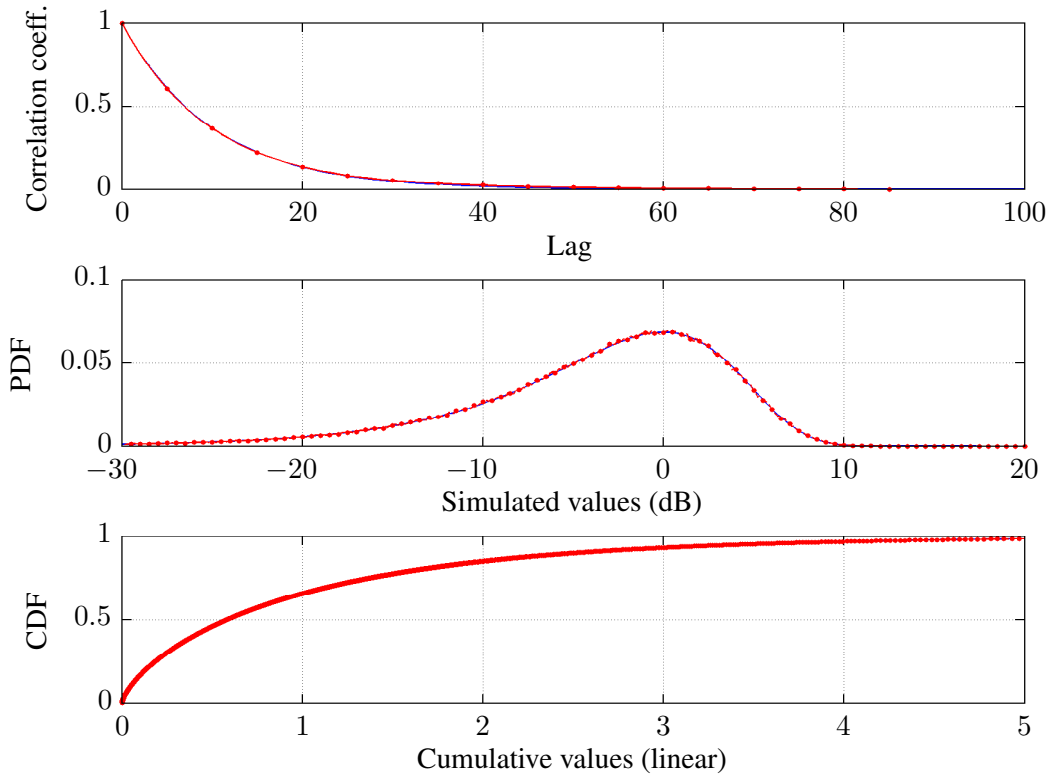


Figure 3: *Comparison between the desired and simulated correlation coefficients for Example 2: (—), exact (---) simulated: (top) one-sided correlation coefficients; (middle) PDF's of the Gamma and (bottom) CDF's of the Gamma.*

4.3 Comparison of Algorithms

In this section, comparisons among the hybrid method, numerically integrated MNLT approach as well as the polynomial MNLT implementation are performed in terms of accuracy and timeliness. In particular, the shape parameter is varied from 0.1 to 10 in steps of 0.1 and the mean is set to 1. 10^4 samples are generated for each realisation and the result is averaged 100 times. It was noted that the polynomial MNLT method does not work efficiently below a shape of 0.3.

The first comparison looked at the accuracy of the measured auto-correlation function by comparing the simulated with the desired. It was found that each of the three methods are very accurate with the maximum RMS error over the 100 iterations being less than 0.002.

The second comparison looks at the relative error in the shape parameter and scale by subtracting the maximum likelihood estimate from the true value and dividing that result by the true value. Figure 4 shows the results with the shape in the top plot and scale in the bottom. The shape parameter estimate is quite good for each method with a small relative error. However, there is a slight increase in the relative error for the polynomial MNLT method when the shape is below 0.7. The scale estimate is also very accurate for all methods when the shape is above 1. For low shape parameters, however, the hybrid method is more accurate.

The third comparison looks at the second, third and fourth order moments of the simulated data. These are also known as the variance, skewness and kurtosis. Figure 5 shows that all three methods produce accurate moments with the numerically integrated MNLT method being less accurate at low shape parameters.

An exact expression for the algorithm complexity is difficult, as the exact implementation details are not known for all algorithms. Instead, the final comparison looks at the mean run times for the three methods. Figure 6 shows that the polynomial MNLT method is the slowest by an order of magnitude, while the performance of the hybrid method varies depending on the shape value. The hybrid method is the fastest for nearly all shape values and approaches the numerically integrated MNLT method as the shape increases. This improvement is significant considering that the inverse incomplete Gamma function in the latter method was coded in C in order to improve computational efficiency, while the hybrid method only uses builtin MATLAB functions.

In summary, the hybrid method is slightly more accurate at low shape values than the other MNLT methods and has a superior mean run time. The polynomial MNLT method is less accurate with low shape parameters and takes a significantly longer time to run.

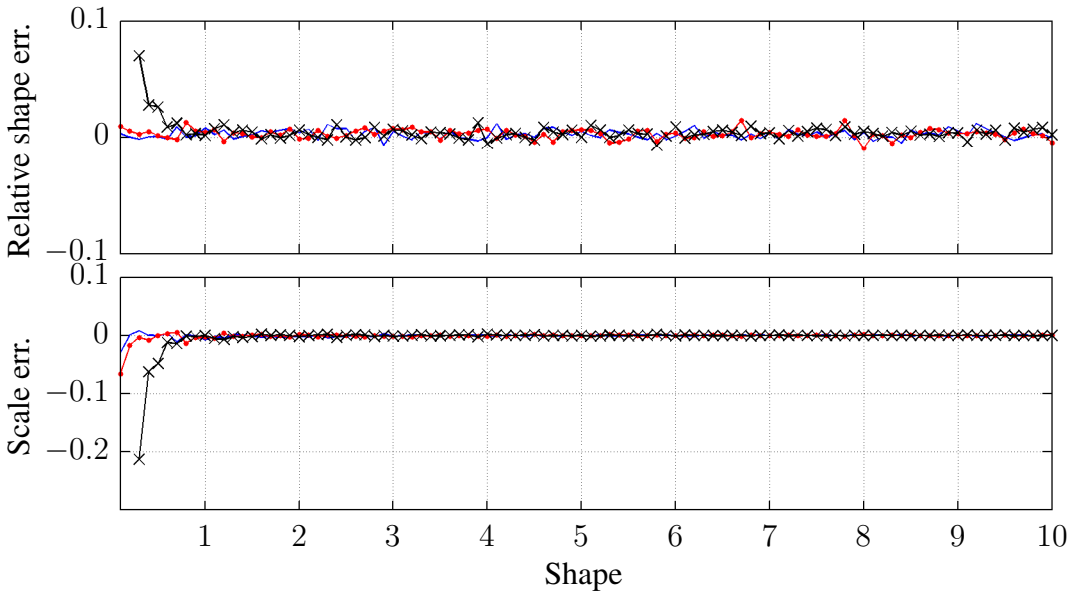


Figure 4: *Parameter estimate error: (—) hybrid, (-×-) MNLT poly., (···) MNLT numerical.*

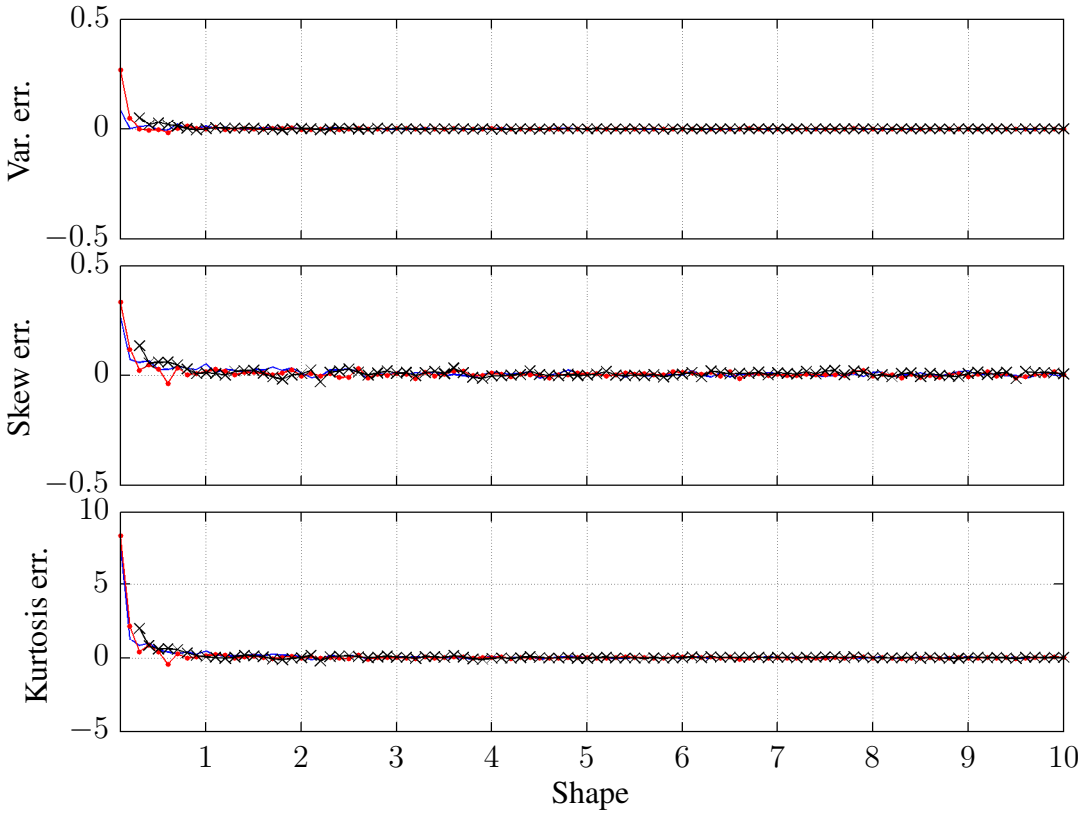


Figure 5: *Moment error: (—) hybrid, (-×-) MNLT poly., (---) MNLT numerical.*

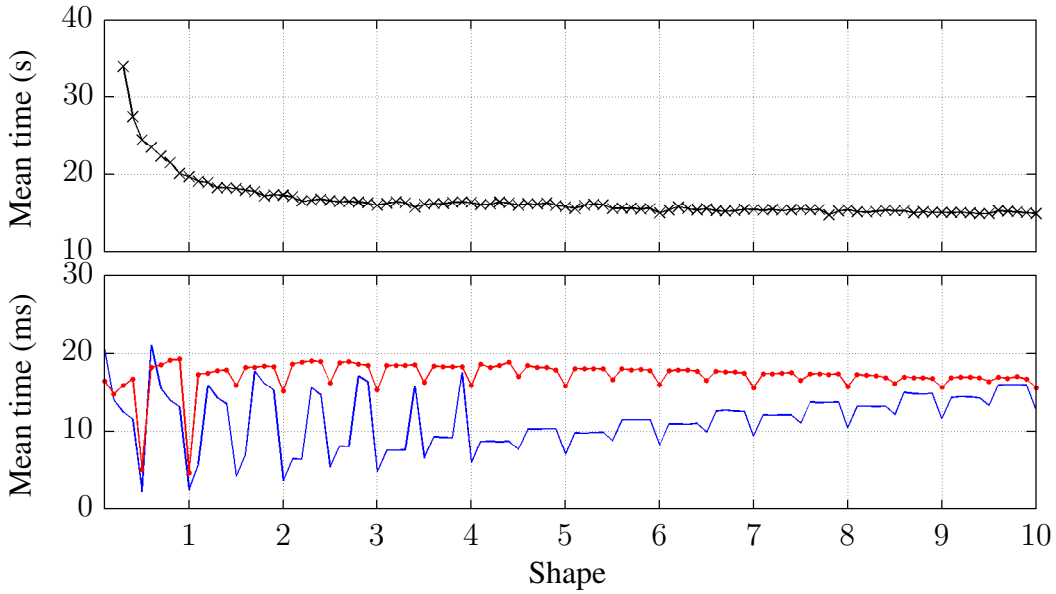
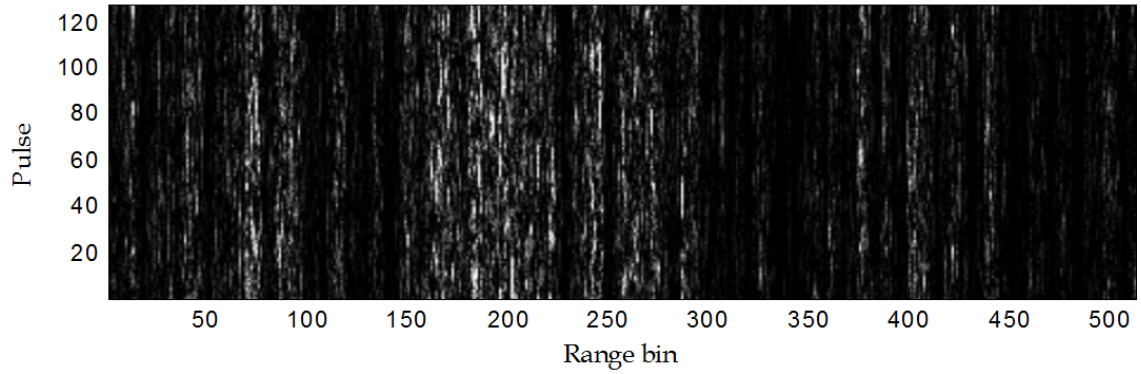


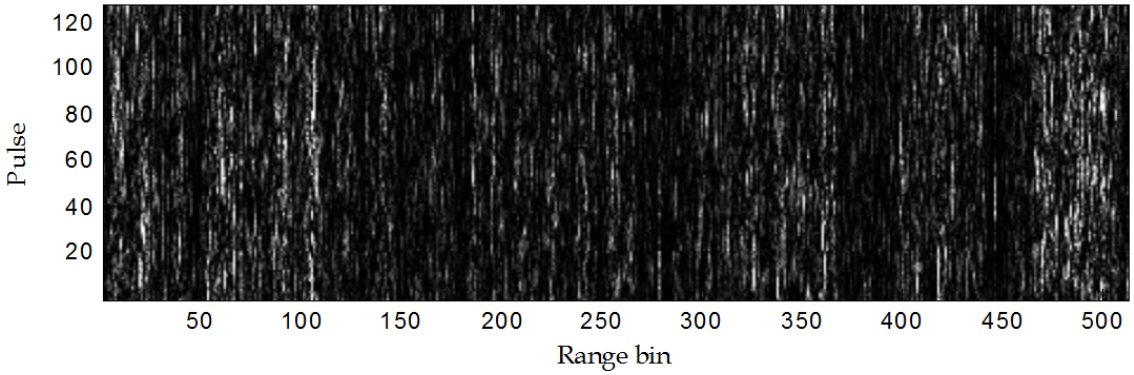
Figure 6: *Mean run time comparison: (—) hybrid, (-×-) MNLT poly., (---) MNLT numerical.*

4.4 K-distribution Example

As a final example, two range-time (pulse) intensity images of correlated K-distributed sea-clutter are shown in Figure 7. The temporal correlation, i.e. the correlation among the fast-varying Gaussian component with respect to pulse, is specified by (33), while the spatial correlation, i.e. the correlation among the slowly-varying Gamma component with respect to range bin is specified by (32).



(a) ν=1.2



(b) ν=3.7

Figure 7: *Range-time (pulse) intensity maps of correlated K-distributed sea-clutter. Both have the same correlation but different shape parameters. Normalised intensity values* ≥ 0.5 *are shown as white in the images.*

5 Conclusion

This report has systematically studied how to generate correlated Gamma processes to support the simulation of sea-clutter for many DSTO projects. In particular, a hybrid method was proposed for generating the desired correlated Gamma samples. The method uses only linear transformations for most values of the shape parameter and correlation, with a non-linear transform required in some cases. The difficulty of the non-linear transform is the mapping of the correlation of the desired output to the correlation of the input, often a Gaussian process. Two MNLT implementations were presented for comparison, one based on numerical integration and the other a polynomial approximation. Performance of the three methods has been evaluated, focusing on the accuracy of the desired correlated process and the timeliness for the realisation. It was found that the hybrid method is the slightly more accurate for small shape parameters. It is also the most computationally efficient and hence the fastest for nearly all shape values. This improvement is significant considering that the numerically integrated MNLT method has the inverse incomplete Gamma function coded in C in order to improve computational efficiency, while the hybrid method only uses builtin MATLAB functions. The drawback of the hybrid method is, however, that it can only handle positive correlations whilst the two MNLT methods are capable of handing both positive and negative correlations.

6 Acknowledgement

The authors thank Dr. Stephen Bocquet from Joint Operations Division of DSTO for his comments on Section 3.1 and providing the efficient MNLT implementation of [Tough & Ward 1999]. His insights into the implementation of this method greatly helped the authors to understand the method. As a formal reviewer, Dr. Bocquet also made valuable comments and pointed out some errors in the draft. Undoubtedly his contribution greatly improved the final form of the report.

Comments from Dr. Brett Haywood of Electronic Warfare and Radar Division of DSTO are also acknowledged.

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Appendix A Temporal and Spatial Correlations of Compound K-Distributed Clutter

The compound K-distribution assumes that the data consists of a fast-varying component modulated by a slowly-varying component. The fast-varying component commonly refers to the speckle component that is a Gaussian random process with zero mean and unit variance. The slowly-varying component also describes the underlying mean or texture with intensity modelled by a Gamma distribution. Care must be taken when estimating the correlations of these components as the slowly-varying component may not always remain constant when estimating the temporal correlation.

A.1 Temporal Correlation

The temporal correlation is present between data samples collected by a pulse train in a coherent processing interval (CPI). Often during the CPI, it can be assumed that the slowly-varying component remains unchanged, i.e., the slowly varying component is fully correlated (this is how the compound K-distributed clutter got its name)[4]. The covariance matrix of the data is written as,

$$\mathbf{M} = E\left\{\mathbf{x}\mathbf{x}^H\right\}, \tag{A1}$$

where expectation is with respect to the pulse for the temporal correlation, and $\mathbf{x} = [x[0], \ldots, x[N-1]]^T$ is an $N \times 1$ vector collected by N pulses. Each measurement may be further written as a product of fast-varying and slowly-varying components, according to the model assumption,

$$x[n] = \sqrt{\tau} x_f[n], \tag{A2}$$

where $x_f[n]$ is the fast-varying component which is a complex Gaussian variable and τ is the underlying mean which is constant in a CPI. Because the two components are independent,

$$\mathbf{M} = E\left\{\mathbf{x}\mathbf{x}^H\right\} = E\left\{\tau\right\} E\left\{\mathbf{x}_f\mathbf{x}_f^H\right\} = \mu\mathbf{M}_f, \tag{A3}$$

where $\mathbf{M}_f = E\left\{\mathbf{x}_f\mathbf{x}_f^H\right\}$ is the covariance matrix of the fast-varying component and μ is the clutter mean. Therefore, the temporal correlation can be estimated using data samples and the correlation of the fast-varying component is just the covariance matrix normalised by its mean. Under the assumption of wide sense stationary, $\mathbf{M}_f$ has a Toeplitz structure of,

$$\mathbf{M}_f = \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{N-1} \\ \rho_1^* & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \rho_1 \\ \rho_{N-1}^* & \cdots & \rho_1^* & 1 \end{bmatrix}, \tag{A4}$$

where $\rho_k = \frac{1}{\mu} E\left\{x[n]x^*[n+k]\right\}, n, k = 0, 1, \ldots, N-1$.

[4] With reference to a maritime search radar, there is high superposition of antenna footprints with respect to successive pulses in a CPI, so the texture of sea-clutter can be assumed to be completely correlated.

A.2 Spatial Correlation (Correlation in Range)

In order to find the correlation of the slowly varying component, the data needs to be manipulated in the intensity (or amplitude) domain rather than the complex (I/Q) domain. Consider the correlation between range bin n and $n+k$, $k \neq 0$[5],

$$E\left\{\sqrt{\tau[n]}x_f[n]\sqrt{\tau[n+k]}x_f^*[n+k]\right\} = E\left\{\sqrt{\tau[n]\tau[n+k]}\right\} E\left\{x_f[n]x_f^*[n+k]\right\}. \quad \text{(A5)}$$

If however, the interval between bin n and bin $n+k$ is greater than the radar range resolution, $E\left\{x_f[n]x_f^*[n+k]\right\} \equiv 0$. Therefore, even if $E\left\{\sqrt{\tau[n]\tau[n+k]}\right\} \neq 0$, its value is not measurable using the I/Q data. The correlation therefore has to be found using the intensity (or amplitude) data. The correlation coefficients of texture, $\tau[n]$ and intensity, $z[n] = x^2[n]$ are given by,

$$\rho_k = \frac{E\left\{\tau[n]\tau[n+k]\right\} - E^2\left\{\tau\right\}}{\text{var}(\tau)}, \qquad k = 0, 1, \ldots, \quad \text{(A6)}$$

$$\chi_k = \frac{E\left\{z[n]z[n+k]\right\} - E^2\left\{z\right\}}{\text{var}(z)}, \qquad k = 0, 1, \ldots, \quad \text{(A7)}$$

with

$$\chi_0 = \rho_0 = 1. \quad \text{(A8)}$$

The next step is to find the relationship between χ_k and ρ_k for $k \neq 0$. Denoting $z = \tau|x_f|^2 = \tau z_f$, gives

$$E\left\{z[n]z[n+k]\right\} = E\left\{\tau[n]\tau[n+k]z_f[n]z_f[n+k]\right\} \quad \text{(A9)}$$

and since the fast-varying and slowly varying components are independent,

$$E\left\{\tau[n]\tau[n+k]z_f[n]z_f[n+k]\right\} = E\left\{\tau[n]\tau[n+k]\right\} E\left\{z_f[n]z_f[n+k]\right\}. \quad \text{(A10)}$$

For $k \neq 0$, the value of $E\left\{z_f[n]z_f[n+k]\right\}$ can be calculated using Isserlis' Theorem [Michalowicz et al. 2009], giving

$$\begin{aligned} E\left\{z_f[n]z_f[n+k]\right\} &= E\left\{x_f[n]x_f^*[n]x_f[n+k]x_f^*[n+k]\right\} \\ &= 1 + E\left\{x_f[n]x_f[n+k]\right\} E\left\{x_f^*[n]x_f^*[n+k]\right\} \qquad \text{for } k \neq 0. \quad \text{(A11)} \\ &\quad + E\left\{x_f[n]x_f^*[n+k]\right\} E\left\{x_f^*[n]x_f[n+k]\right\} \end{aligned}$$

If the interval between bin n and bin $n+k$ is greater than the radar range resolution, the last two items of (A11) become zero and $E\left\{z_f[n]z_f[n+k]\right\} = 1$, simplifying the above correlation to

$$E\left\{\tau[n]\tau[n+k]\right\} = E\left\{z[n]z[n+k]\right\} \qquad \text{for } k \neq 0. \quad \text{(A12)}$$

[5] In order to simplify symbols, the same index notation is used to represent either temporal series (the index refers to pulse numbers) or spatial series (the index refers to range bin number). There should be no confusion under the context.

Combining (A6), (A7) and (A12) and noticing $E\{\tau\} = E\{z\} = \mu$, then gives

$$\chi_k = \rho_k \frac{\text{var}(\tau)}{\text{var}(z)}, \qquad k = 1, 2 \ldots . \tag{A13}$$

Equations (A8) and (A13) indicate that once the correlation of z or τ is known, the correlation of the other can be determined. The texture τ is Gamma distributed, and the intensity z is single-look or multi-look K-distributed, with variances given respectively as,

$$\text{var}(\tau) = \mu^2/\nu, \tag{A14}$$

$$\text{var}(z) = \frac{\nu + N + 1}{N\nu}\mu^2, \tag{A15}$$

where N is the number of multi-looks. Therefore, due to the effect of the fast-varying component that is uncorrelated and randomly varying, the correlation of the intensity z is generally weaker than the correlation of the texture τ. Only if the number of multi-looks reaches infinity, will the fluctuations of the fast-varying component disappear (averaged to its mean value for every range bin) and the two correlations become identical.

The final relationship is therefore found by inserting (A14) and (A15) into (A13), giving,

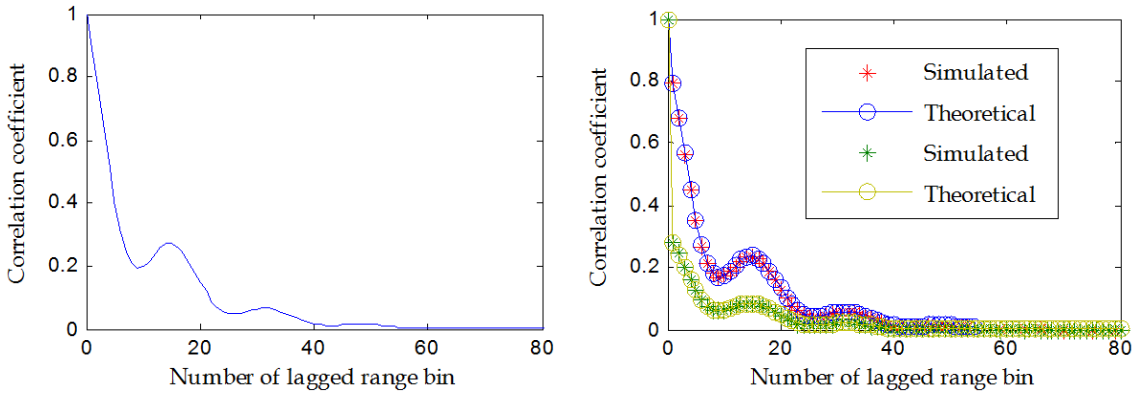
$$\chi_k = \rho_k \frac{N}{\nu + N + 1}, \qquad k = 1, 2 \ldots . \tag{A16}$$

The correctness of the above derivation is confirmed by Monte Carlo simulation where the slowly varying Gamma component has a shape parameter of $\nu = 1.2$ and a correlation coefficient of,

$$\rho_k = [0.7 + 0.3\cos(0.12\pi\, k)]\, e^{-k/12}, \qquad k = 0, 1, \ldots . \tag{A17}$$

The multi-look K distributed data was then generated by modulating the uncorrelated multi-look fast-varying Gaussian component with the correlated Gamma. The correlation coefficient for the simulated multi-look K data was regressed and compared to the theoretical value. Figure A.2 shows a comparison between the ideal and theoretical correlation for the single look case and with 16 looks. It can be seen that the simulated results match the theory with the multi-look averaging reducing the variance of fast-varying component. As a result, $< |x|^2 > \approx 1$ and hence the correlation approaches the correlation of texture. For the single-look case, oscillation of the uncorrelated fast-component greatly reduces the overall correlation.

For over-sampled range data, when the range interval between bin n and bin $n + k$ ($k \neq 0$) is smaller than the range resolution, the fast-varying component is also correlated in range, resulting in $E\{z_f[n] z_f[n + k]\} > 1$ (see (A11)). The correlation of the intensity z will then be jointly contributed by the fast-varying component and the slowly-varying component.



(a) Correlation of gamma (b) correlation of K

Figure A1: *The desired correlation of the (a) Gamma component (16 looks) and (b) the correlation of the multi-look K data for: (-○-, -∗-) 1 look and (-○-, -∗-) 16 looks.*

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION DOCUMENT CONTROL DATA	1. CAVEAT/PRIVACY MARKING

2. TITLE	3. SECURITY CLASSIFICATION
Generating Correlated Gamma Sequences for Sea-Clutter Simulation	Document (U) Title (U) Abstract (U)
4. AUTHORS	5. CORPORATE AUTHOR
Yunhan Dong, Luke Rosenberg and Graham Weinberg	Defence Science and Technology Organisation PO Box 1500 Edinburgh, South Australia 5111, Australia

6a. DSTO NUMBER	6b. AR NUMBER	6c. TYPE OF REPORT	7. DOCUMENT DATE
DSTO–TR–2688	015-270	Technical Report	March 2012

8. FILE NUMBER	9. TASK NUMBER	10. TASK SPONSOR	11. No. OF PAGES	12. No. OF REFS
2012/1021438/1	AIR7000	DGAD	24	16

13. URL OF ELECTRONIC VERSION	14. RELEASE AUTHORITY
`http://www.dsto.defence.gov.au/publications/scientific.php`	Chief, Electronic Warfare and Radar Division

15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT

Approved for Public Release

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16. DELIBERATE ANNOUNCEMENT

No Limitations

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18. DSTO RESEARCH LIBRARY THESAURUS

Radar, sea-clutter, simulation, correlation

19. ABSTRACT

This report presents a hybrid method for simulating sequences of correlated Gamma random variables for modelling sea clutter, using a combination of linear and/or non-linear transforms. Depending on the shape parameter, this method minimises the use of non-linear transformations. Mathematically the method is simpler than its counterpart methods which leads to a quicker simulation run time. Two memoryless non-linear transform (MNLT) approaches are also studied with comparative results showing that the hybrid approach is more computationally efficient and slightly more accurate for low shape parameters. The drawback of the proposed method is, however, that it can only handle positive correlations whilst the two MNLT methods are capable of handling both positive and negative correlations.