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Integrated Navigation, Guidance, and Control of Missile Systems: 2-D Dynamic Models

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ABSTRACT

In this report mathematical models (2-D Azimuth and Elevation Planes) for multi-party engagement kinematics are derived suitable for developing, implementing and testing modern missile guidance systems. The models developed here are suitable for both conventional and more advanced optimal intelligent guidance schemes including those that arise out of the differential game theory. These models accommodate changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces.

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Executive Summary

In the past, linear kinematics models have been used for development and analysis of guidance laws for missile/target engagements. These models were developed in fixed axis systems under the assumption that the engagement trajectory does not vary significantly from the collision course geometry. While these models take into account autopilot lags and acceleration limits, the guidance commands are applied in fixed axis, and ignore the fact that the missile/target attitude may change significantly during engagement. This latter fact is particularly relevant in cases of engagements where the target implements evasive manoeuvres, resulting in large variations of the engagement trajectory from that of the collision course. The linearised models are convenient for deriving guidance laws (in analytical form), however, the study of their performance characteristics still requires a non-linear model that incorporates changes in body attitudes and implements guidance commands in body axis rather than the fixed axis. In this report, azimuth and elevation plane mathematical models for multi-party engagement kinematics are derived suitable for developing, implementing and testing modern missile guidance systems. The models developed here are suitable for both conventional and more advanced optimal intelligent guidance, particularly those based on the 'game theory' guidance techniques. These models accommodate changes in vehicle body attitude and other non-linear effects, such as, limits on lateral acceleration and aerodynamic forces. The models presented in this report will be found suitable for computer simulation and analysis of multi-party engagements.

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Nomenclature

$(\mathbf{i},\mathbf{j})$:	number of interceptors (pursuers) and targets (evaders) respectively.
(x_i, y_i, z_i):	are **x** , **y**, **z**-positions respectively of vehicle **i** in fixed axis.
(u_i, v_i, w_i):	are **x, y, z**-velocities respectively of vehicle **i** in fixed axis.
$(a_{x_i}, a_{y_i}, a_{z_i})$:	are **x, y, z**-accelerations respectively of vehicle **i** in fixed axis.
(x_{ji}, y_{ji}, z_{ji}):	are **x, y, z**-positions respectively of vehicle **j** w.r.t **i** in fixed axis.
(u_{ji}, v_{ji}, w_{ji}):	are **x, y, z**-velocities respectively of vehicle **j** w.r.t **i** in fixed axis.
$(a_{x_{ji}}, a_{y_{ji}}, a_{z_{ji}})$:	are **x, y, z**-accelerations respectively of vehicle **j** w.r.t **i** in fixed axis.
$(\underline{x}_i, \underline{u}_i, \underline{a}_i)$:	are **x** ,**y**-position, velocity and acceleration vectors of vehicle **i** in fixed axis.
$(\underline{x}_{ji}, \underline{u}_{ji}, \underline{a}_{ji})$:	are **x, y**- relative position, velocity and acceleration vectors of vehicle **j** w.r.t **i** in fixed axis.
$(\underline{y}_i, \underline{v}_i, \underline{c}_i)$:	are **x, z**-position, velocity and acceleration vectors of vehicle **i** in fixed axis.
$(\underline{y}_{ji}, \underline{v}_{ji}, \underline{c}_{ji})$:	are **x, z**- relative position, velocity and acceleration vectors of vehicle **j** w.r.t **i** in fixed axis.
R_{ji} :	separation range of vehicle **j** w.r.t **i** in fixed axis.
$V_{c_{ji}}$:	closing velocity of vehicle **j** w.r.t **i** in fixed axis.
$\lambda_{ji}, \gamma_{ji}$:	are line-of-sight angle (LOS) of vehicle **j** w.r.t **i** in yaw and pitch planes respectively.
$(a^b_{x_i}, a^b_{y_i}, a^b_{z_i})$:	**x, y, z**-accelerations respectively achieved by vehicle **i** in body axis.
$(a^b_{x_{i_d}}, a^b_{y_{i_d}}, a^b_{z_{i_d}})$:	**x, y, z**-accelerations respectively demanded by vehicle **i** in body axis.
$\underline{a}^b_i, \underline{c}^b_i$:	is the achieved missile acceleration vector in body axis.
$\underline{a}^b_{i_d}, \underline{c}^b_{i_d}$:	is the demanded missile acceleration vector in body axis.
ψ_i, θ_i :	are yaw and pitch body (Euler) angles respectively of the **i**th vehicle w.r.t the fixed axis.
$[T^f_b]_i$:	is the transformation matrix from body axis to fixed axis.
V_i :	is the velocity of vehicle **i**.
τ_{x_i} :	autopilot's longitudinal time-constant for vehicle **i**.
τ_{y_i}, τ_{z_i} :	autopilot's lateral time-constant for vehicle **i**.

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1. Introduction

In the past [1, 2] linear kinematics models have been used for development and analysis of guidance laws for missile/target engagements. These models were developed in fixed axis under the assumption that the engagement trajectory does not vary significantly from the collision course geometry. While these models take into account autopilot lags and acceleration limits, the guidance commands are applied in fixed axis, and ignore the fact that the missile/target attitude may change significantly during engagement. This latter fact is particularly relevant in cases of engagements where the target implements evasive manoeuvres, resulting in large variations of the engagement trajectory from that of the collision course [3]. The linearised models are convenient for deriving guidance laws (in analytical form), however, the study of their performance characteristics still requires a non-linear model that incorporates changes in body attitudes and implements guidance commands in body axis rather than the fixed axis.

In this report, mathematical models for multi-party engagement kinematics are derived suitable for developing, implementing and testing modern missile guidance systems. The models developed here are suitable for both conventional and more advanced optimal intelligent guidance, particularly those based on the 'game theory' guidance techniques. The models accommodate changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces. Body incidence is assumed to be small and is neglected. The models presented in this report will be found suitable for computer simulation and analysis of multi-party engagements. Sections 2-4 of this report considers in some detail the derivation of engagement dynamics in azimuth (Az) plane. Subsequent sections consider the engagement dynamics in elevation (El) plane and perhaps more relevant to trajectories that are typical of glide vehicles.

2. Development of Azimuth Plane Engagement Kinematics Model

2.1 Translational Kinematics for Multi-Vehicle Engagement

A typical 2-vehicle engagement geometry is shown in Figure 1, we define the following variables:

$(\mathbf{x_i}, \mathbf{y_i})$**:** are **x** ,**y**-positions respectively of vehicle **i** in fixed axis.

$(\mathbf{u_i}, \mathbf{v_i})$**:** are **x, y**-velocities respectively of vehicle **i** in fixed axis.

$(\mathbf{a_{x_i}}, \mathbf{a_{y_i}})$**:** are **x, y**-accelerations respectively of vehicle **i** in fixed axis.

The above variables are functions of time $\mathbf{t}$. Then the motion of vehicle for **n** interceptors and **m** targets $(\mathbf{i : i = 1,2,...,n + m})$ (i.e. position, velocity and acceleration) in fixed (e.g. inertial) axis is given by the following differential equations:

$$\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{x_i} = \mathbf{u_i} \qquad (2.1)$$

$$\frac{d}{dt}y_i = v_i \tag{2.2}$$

$$\frac{d}{dt}u_i = a_{x_i} \tag{2.3}$$

$$\frac{d}{dt}v_i = a_{y_i} \tag{2.4}$$

For multiple vehicles **i, j** engagement, we define the relative variables (states) for $(i : i = 1,2,\ldots,n;\ j = 1,2,\ldots,m;\ j \neq i)$ as follows:

$x_{ji} = x_j - x_i$: **x**-position of vehicle **j** w.r.t **i** in fixed axis.

$y_{ji} = y_j - y_i$: **y**-position of vehicle **j** w.r.t **i** in fixed axis.

$u_{ji} = u_j - u_i$: **x**-velocity of vehicle **j** w.r.t **i** in fixed axis.

$v_{ji} = v_j - v_i$: **y**-velocity of vehicle **j** w.r.t **i** in fixed axis.

$a_{x_{ji}} = a_{x_j} - a_{x_i}$: **x**-acceleration of vehicle **j** w.r.t **i** in fixed axis.

$a_{y_{ji}} = a_{y_j} - a_{yi}$: **y**-acceleration of vehicle **j** w.r.t i in fixed axis.

2.1.1 Vector/Matrix Representation

We can write equations (2.1)-(2.4), in vector notation as:

$$\frac{d}{dt}\underline{x}_i = \underline{u}_i \tag{2.5}$$

$$\frac{d}{dt}\underline{u}_i = \underline{a}_i \tag{2.6}$$

Where:

$\underline{x}_i = [x_i \quad y_i]^T$: is the position vector of vehicle **i** in fixed axis.

$\underline{u}_i = [u_i \quad v_i]^T$: is the velocity vector of vehicle **i** in fixed axis.

$\underline{a}_i = [a_{x_i} \quad a_{y_i}]^T$: is the target acceleration vector of vehicle **i** in fixed axis.

Similarly, we can write the relative kinematics vector equations as:

$$\frac{d}{dt}\underline{x}_{ji} = \underline{u}_{ji} \tag{2.7}$$

$$\frac{d}{dt}\underline{u}_{ji} = \underline{a}_j - \underline{a}_i \tag{2.8}$$

Where:

$\underline{x}_{ji} = [x_{ji} \quad y_{ji}]^T$: position vector of vehicle **j** w.r.t **i** in fixed axis.

$\underline{u}_{ji} = [u_{ji} \quad v_{ji}]^T$: velocity vector of vehicle **j** w.r.t **i** in fixed axis.

$\underline{a}_{ji} = [a_{x_{ji}} \quad a_{y_{ji}}]^T = \underline{a}_j - \underline{a}_i$: acceleration vector of vehicle **j** w.r.t **i** in fixed axis.

Note: The above formulation admits consideration of engagement where one particular vehicle (interceptor) is fired at another single vehicle (target). In other words we consider one-against-one in a scenario consisting of many vehicles. This consideration can be extended to one-against-many if, for example, **i** takes on a single value and **j** is allowed to take on a number of different values.

2.2 Constructing Relative Sightline (LOS) Angles and Rates - (Rotational Kinematics)

The separation range $\mathbf{R_{ji}}$ between vehicles **j** w.r.t **i** is given by:

$$\mathbf{R_{ji}} = \left(\mathbf{x_{ji}}^2 + \mathbf{y_{ji}}^2\right)^{\frac{1}{2}} = \left(\underline{\mathbf{x}}_{\mathbf{ji}}{}^{\mathbf{T}}\,\underline{\mathbf{x}}_{\mathbf{ji}}\right)^{\frac{1}{2}} \tag{2.9}$$

The range rate $\dot{\mathbf{R}}_{\mathbf{ji}}$ is given by:

$$\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{R_{ji}} = \dot{\mathbf{R}}_{\mathbf{ji}} = \frac{\mathbf{x_{ji}u_{ji}} + \mathbf{y_{ji}v_{ji}}}{\mathbf{R_{ji}}} = \frac{\left(\underline{\mathbf{x}}_{\mathbf{ji}}{}^{\mathbf{T}}\,\underline{\mathbf{u}}_{\mathbf{ji}}\right)}{\mathbf{R_{ji}}} \tag{2.10}$$

The Closing Velocity $\mathbf{V_{c_{ji}}}$ is given by:

$$\mathbf{V_{c_{ji}}} = -\dot{\mathbf{R}}_{\mathbf{ji}} \tag{2.11}$$

The sightline angle $\lambda_{\mathbf{ji}}$ of **j** w.r.t **i** is given by:

$$\mathbf{tan}\,\lambda_{\mathbf{ji}} = \frac{\mathbf{y_{ji}}}{\mathbf{x_{ji}}} \tag{2.12}$$

Differentiating both sides of the equation and simplifying, we get:

$$\frac{\mathbf{d}}{\mathbf{dt}}\lambda_{\mathbf{ji}} = \dot{\lambda}_{\mathbf{ji}} = \frac{\dot{\mathbf{y}}_{\mathbf{ji}}\mathbf{x_{ji}}}{\mathbf{R_{ji}}^2} - \frac{\mathbf{y_{ji}}\dot{\mathbf{x}}_{\mathbf{ji}}}{\mathbf{R_{ji}}^2} = \frac{\mathbf{x_{ji}v_{ji}} - \mathbf{y_{ji}u_{ji}}}{\mathbf{R_{ji}}^2} = \frac{\underline{\mathbf{x}}_{\mathbf{ji}}{}^{\mathbf{T}}[\mathbf{J}]\underline{\mathbf{u}}_{\mathbf{ji}}}{\mathbf{R_{ji}}^2} \tag{2.13}$$

Where:

$$[\mathbf{J}] = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$$

Note that an approximation to (2.13) is sometimes used, although not recommended for simulation purposes, based on the assumptions that the engagement geometry does not deviate significantly from the collision course; in this case:
$\mathbf{x_{ji}} \approx \mathbf{R_{ji}} = \mathbf{V_{c_{ji}}T_{go}}$; $\dot{\mathbf{x}}_{\mathbf{ji}} \approx \dot{\mathbf{R}}_{\mathbf{ji}} = -\mathbf{V_{c_{ji}}}$; $\mathbf{T_{go}} = (\mathbf{t_f} - \mathbf{t})$**:** is the time-to-go.

Substituting this in equation (2.13) gives us:

$$\dot{\lambda}_{\mathbf{ji}} = \frac{\mathbf{1}}{\mathbf{V_{c_{ji}}}}\left(\frac{\dot{\mathbf{y}}_{\mathbf{ji}}}{\mathbf{T_{go}}} + \frac{\mathbf{y_{ji}}}{\mathbf{T_{go}}^2}\right) \tag{2.14}$$

The measurements $\hat{\dot{\lambda}}_{ji}$ obtained from the seeker that are used to construct the guidance commands are given by:

$$\hat{\dot{\lambda}}_{ji} = \dot{\lambda}_{ji} + \Delta\dot{\lambda}_{ji} \tag{2.15}$$

Where:

$\Delta\dot{\lambda}_{ji}$ **:** seeker LOS rate measurement error.

The above relationships (2.9)-(2.15) will also be referred to as the seeker model.

2.3 Vehicle Navigation Model

Let us define the following:

$\mathbf{a^b_{x_i}}$ **:** **x**-acceleration achieved by vehicle **i** in its body axis.

$\mathbf{a^b_{y_i}}$ **:** **y**-acceleration achieved by vehicle **i** in its body axis.

The transformation from fixed to body axis is given by (see Figure 2):

$$\begin{bmatrix} \mathbf{a^b_{x_i}} \\ \mathbf{a^b_{y_i}} \end{bmatrix} = \begin{bmatrix} \cos\psi_i & \sin\psi_i \\ -\sin\psi_i & \cos\psi_i \end{bmatrix} \begin{bmatrix} \mathbf{a_{x_i}} \\ \mathbf{a_{y_i}} \end{bmatrix} = \left[\mathbf{T^b_f}\right]_i \begin{bmatrix} \mathbf{a_{x_i}} \\ \mathbf{a_{y_i}} \end{bmatrix} \tag{2.16}$$

In vector/matrix notation this equation may be written as:

$$\underline{\mathbf{a}}^b_i = \left[\mathbf{T^b_f}\right]_i \underline{\mathbf{a}}_i \tag{2.17}$$

$$\underline{\mathbf{a}}_i = \left[\mathbf{T^f_b}\right]_i \underline{\mathbf{a}}^b_i \tag{2.18}$$

Where:

ψ_i : is the yaw (Euler) angle of the **i**th vehicle w.r.t the fixed axis. It is assumed that the body orientation ψ_i changes during the engagement.

$\underline{\mathbf{a}}^b_i = \left[\mathbf{a^b_{x_i}} \quad \mathbf{a^b_{y_i}}\right]^T$ **:** is acceleration vector of vehicle **i** in its body axis.

$\left[\mathbf{T^f_b}\right]_i = \left[\mathbf{T^b_f}\right]^T_i = \left[\mathbf{T^b_f}\right]^{-1}_i$ **:** is the transformation matrix from body axis to fixed axis.

$\left[\mathbf{T^f_b}\right]_i = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}$ **:** is the (direction cosine) transformation matrix from body to fixed axis.

The vehicle velocity is given by:

$$\mathbf{V_i} = \left(\mathbf{u_i}^2 + \mathbf{v_i}^2\right)^{\frac{1}{2}} = \left(\underline{\mathbf{u}}_i^T \underline{\mathbf{u}}_i\right)^{\frac{1}{2}} \tag{2.19}$$

Now the flight path angle (=angle that the velocity vector makes with the fixed axis) is given by:

$$\tan(\psi_i - \beta_i) = \frac{v_i}{u_i} \tag{2.20}$$

Where:

$\beta_i = \tan^{-1}\left(\frac{v_b}{u_b}\right)$: is the azimuth body incidence (side-slip angle), $\{u_b, v_b\}$ are body axis velocities (Figure A1.1).

Differentiating both sides of equation (2.20) and simplifying we get:

$$\dot{\psi}_i - \dot{\beta}_i = \frac{\dot{v}_i u_i}{V_i^2} - \frac{v_i \dot{u}_i}{V_i^2} = \frac{u_i a_{y_i} - v_i a_{x_i}}{V_i^2} = \frac{\underline{u}_i^T [J] \underline{a}_i}{V_i^2} \tag{2.21}$$

Assuming $\beta_i, \dot{\beta}_i$ remain small ($u_b >> v_b$; see also Appendix A), then we may write:

$$\frac{d}{dt}\psi_i = \frac{\dot{v}_i u_i}{V_i^2} - \frac{v_i \dot{u}_i}{V_i^2} = \frac{u_i a_{y_i} - v_i a_{x_i}}{V_i^2} = \frac{\underline{u}_i^T [J] \underline{a}_i}{V_i^2} \tag{2.22}$$

2.4 Vehicle Autopilot Dynamics

Assuming a first order lag for the autopilot, we may write for vehicle i:

$$\frac{d}{dt} a^b_{x_i} = -\tau_{x_i} a^b_{x_i} + \tau_{x_i} a^b_{x_{i_d}} \tag{2.23}$$

$$\frac{d}{dt} a^b_{y_i} = -\tau_{y_i} a^b_{y_i} + \tau_{y_i} a^b_{y_{i_d}} \tag{2.24}$$

In vector/matrix notation equations (2.23), (2.24) may be written as:

$$\frac{d}{dt}\underline{a}^b_i = [-\Lambda_i]\underline{a}^b_i + [\Lambda_i]\underline{a}^b_{i_d} \tag{2.25}$$

Where:

τ_{x_i} : Vehicle **i** autopilot's longitudinal time-constant.

τ_{y_i} : Vehicle **i** autopilot's lateral time-constant.

$[\Lambda_i] = \begin{bmatrix} \tau_{x_i} & 0 \\ 0 & \tau_{y_i} \end{bmatrix}$

$a^b_{x_{i_d}}$: **x**-acceleration demanded by vehicle **i** in its body axis.

$a^b_{y_{i_d}}$: **y**-acceleration demanded by vehicle **i** in its body axis.

$\underline{a}^b_{i_d} = \begin{bmatrix} a^b_{x_{i_d}} & a^b_{y_{i_d}} \end{bmatrix}^T$: is the demanded missile acceleration (command input) vector in body axis.

Remarks: Detailed consideration of the effects of the aerodynamic forces is contained in Appendix A. Generally, the longitudinal acceleration $\mathbf{a^b_{x_{i_d}}} = \frac{\delta T_i - \delta D_i}{m_i}$ of a missile is not varied in response to the guidance commands and may be assumed to be zero. However, the nominal values: $\bar{\mathbf{a}}^{\mathbf{b}}_{\mathbf{x_i}} = \frac{(\bar{T} - \bar{D})}{m}$ will change due to changes in flight conditions and needs to be included in the simulation model; this is shown in the block diagram Figure 6. The limits on the lateral acceleration can be implemented as: $\left\| \mathbf{a^b}_{\mathbf{y_{i_d}}} \right\| \le \mu\, \mathbf{a_{y_{max}}}$ (see Appendix A).

3. Guidance Laws

3.1 Proportional Navigation (PN) Guidance

There are at least three versions of PN guidance laws that the author is aware of; these are (for vehicle **i** - the pursuer against an intercept (target) vehicle **j** – the evader):

3.1.1. Version 1 (PN-1):

This implementation is based on the principle that the demanded body rate of the attacker **i** is proportional to LOS rate to the target **j** (see Figure 1); that is:

$$\dot{\psi}_{i_d} = N\dot{\lambda}_{ji} \tag{3.1}$$

Where: $\mathbf{N}$ **:** is the navigation constant. Thus the demanded attacker lateral acceleration is given by:

$$a^b_{y_{i_d}} = V_i \dot{\psi}_{i_d} = NV_i \dot{\lambda}_{ji} \tag{3.2}$$

$$a^b_{x_{i_d}} = \frac{\delta T_i - \delta D_i}{m_i} \tag{3.3}$$

3.1.2. Version 2 (PN-2):

This implementation is based on the principle that the demanded lateral acceleration of the attacker **i** is proportional to the acceleration perpendicular (normal) to the LOS rate to the target **j**. Now the acceleration normal to the LOS is given by:

$$a_{n_{ji}} = V_{c_{ji}} \dot{\lambda}_{ji} \tag{3.4}$$

➔

$$a^b_{y_{i_d}} = Na_{n_{ji}} = NV_{c_{ji}} \dot{\lambda}_{ji} \tag{3.5}$$

$$a^b_{x_{i_d}} = \frac{\delta T_i - \delta D_i}{m_i} \tag{3.6}$$

Where:

$\mathbf{V_{c_{ji}}} = -\dot{\mathbf{R}}_{\mathbf{ji}}$ **:** is the '*closing velocity*' between the attacker and the target.

$\mathbf{a_{n_{ji}}}$ **:** is the acceleration normal to the LOS.

3.1.3. Version 3 (PN-3):

This type of guidance law is similar to version 2 except that the normal LOS acceleration is resolved along the lateral direction to the attacker body axis first before applying the proportionality principal. Thus, we have:

$$\mathbf{a_{n_{ji}}} = \mathbf{V_{c_{ji}}} \dot{\lambda}_{\mathbf{ji}} \tag{3.7}$$

giving us the guidance commands:

$$\mathbf{a^b_{y_{i_d}}} = \mathbf{N a_{n_{ji}}} \cos\left(\psi_{\mathbf{i}} - \lambda_{\mathbf{ji}}\right) = \mathbf{N V_{c_{ji}}} \cos\left(\psi_{\mathbf{i}} - \lambda_{\mathbf{ji}}\right) \dot{\lambda}_{\mathbf{ji}} \tag{3.8}$$

$$\mathbf{a^b_{x_{i_d}}} = \frac{\delta \mathbf{T_i} - \delta \mathbf{D_i}}{\mathbf{m_i}} \tag{3.9}$$

3.2 Augmented Proportional Navigation (APN) Guidance

Finally, a variation of the PN guidance law is the APN that can be implemented as follows:

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} = (\mathbf{PNG}) + \mathbf{N}' \left[\mathbf{T^b_f}\right]_{\mathbf{i}} \underline{\mathbf{a}}_{\mathbf{j}} \tag{3.10}$$

Where:

$\mathbf{N}'$ **:** is the (target) acceleration navigation constant

$(\mathbf{PNG})$ **:** is the proportional navigation guidance law given in (3.1)-(3.7)

Remarks

- Seeker errors can be introduced by replacing $\dot{\lambda}_{\mathbf{ji}}$ by $\dot{\hat{\lambda}}_{\mathbf{ji}}$ in the guidance laws above.
- In certain engagement geometries $\mathbf{V_{c_{ji}}}$, $\mathbf{cos(..)}$, $\mathbf{sin(..)}$ and $\dot{\lambda}_{\mathbf{ji}}$ terms in the above equations may become zero prior to termination of the engagements, particularly for manoeuvring targets, and it may become necessary to apply additional disturbances to achieve successful intercept.

4. Overall Azimuth Plane State Space Model

The overall non-linear state space model (e.g. for APN guidance) that can be used for sensitivity studies and for non-linear or Monte-Carlo analysis is given below:

$$\frac{\mathbf{d}}{\mathbf{dt}} \underline{\mathbf{x}}_{\mathbf{ji}} = \underline{\mathbf{u}}_{\mathbf{ji}} \tag{4.1}$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \underline{\mathbf{u}}_{\mathbf{ji}} = \left[\mathbf{T^f_b}\right]_{\mathbf{j}} \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{j}} - \left[\mathbf{T^f_b}\right]_{\mathbf{i}} \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} \tag{4.2}$$

$$\frac{d}{dt}\lambda_{ji} = \frac{\underline{x}_{ji}^{T}[J]\underline{u}_{ji}}{\underline{x}_{ji}^{T}\underline{x}_{ji}} \tag{4.3}$$

$$\underline{a}_{i_d}^{b} = \left\{(PNG) + N'\left[T_f^b\right]_i \underline{a}_j\right\} \tag{4.4}$$

$$\frac{d}{dt}\underline{a}_i^b = [-\Lambda_i]\underline{a}_i^b + [\Lambda_i]\underline{a}_{i_d}^b \tag{4.5}$$

$$\frac{d}{dt}\psi_i = \dot{\psi}_i = \frac{\underline{u}_i^{T}[J]\underline{a}_i}{\underline{u}_i^{T}\underline{u}_i} \tag{4.6}$$

The overall state space model that can be implemented on the computer is given in Table 4.1, and the block-diagram is shown in Figure 6.

5. Extension to Elevation Plane Engagement Model

The axis transformation diagram is shown in Figure 4. We shall point out the fact that the order of rotation is in the order (yaw, pitch and roll), i.e. $(\psi \to \theta \to \phi)$. Of course for a 2-D yaw–pitch decoupled model $\phi = 0$, however, we shall continue to follow this convention for the derivation given in this report. Figures 2 and 5 depict the yaw and pitch transformation diagrams separately. It will be noted, therefore, the key differences in the yaw and the pitch derivation is the transformation matrix, and the steady-state aerodynamic forces acting in the two planes.

5.1 Translational Kinematics for Multi-Vehicle Engagement

The development of the elevation (El) plane model follows closely the methodology used in the development of the Az model; we define the following variables:

(x_i, z_i): are **x** ,**z**-positions respectively of vehicle **i** in fixed axis.
(u_i, w_i): are **x, z**-velocities respectively of vehicle **i** in fixed axis.
(a_{x_i}, a_{z_i}): are **x, z**-accelerations respectively of vehicle **i** in fixed axis.

The motion of vehicle for **n** interceptors and **m** targets $(i : i = 1,2,\ldots,n+m)$ (i.e. position, velocity and acceleration) in fixed (e.g. inertial) axis is given by the following differential equations:

$$\frac{d}{dt}x_i = u_i \tag{5.1}$$

$$\frac{d}{dt}z_i = w_i \tag{5.2}$$

$$\frac{d}{dt}u_i = a_{x_i} \tag{5.3}$$

$$\frac{d}{dt}w_i = a_{z_i} \tag{5.4}$$

For multiple vehicles **i, j** engagement, we define the relative variables (states) for $(\mathbf{i : i = 1,2,...,n; j = 1,2,...,m})$ as follows:

$\mathbf{x_{ji} = x_j - x_i}$ **:** **x**-position of vehicle **j** w.r.t **i** in fixed axis.

$\mathbf{z_{ji} = z_j - z_i}$ **:** **z**-position of vehicle **j** w.r.t **i** in fixed axis.

$\mathbf{u_{ji} = u_j - u_i}$ **:** **x**-velocity of vehicle **j** w.r.t **i** in fixed axis.

$\mathbf{w_{ji} = w_j - w_i}$ **:** **z**-velocity of vehicle **j** w.r.t **i** in fixed axis.

$\mathbf{a_{x_{ji}} = a_{x_j} - a_{x_j}}$ **:** **x**-acceleration of vehicle **j** w.r.t **i** in fixed axis.

$\mathbf{a_{z_{ji}} = a_{z_j} - a_{zi}}$ **:** **z**-acceleration of vehicle **j** w.r.t **i** in fixed axis.

5.1.1. Vector/Matrix Representation

We can write equations (5.1)-(5.4), in vector notation as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{y}}_{\mathbf{i}} = \underline{\mathbf{v}}_{\mathbf{i}} \tag{5.5}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{v}}_{\mathbf{i}} = \underline{\mathbf{c}}_{\mathbf{i}} \tag{5.6}$$

Similarly, we can write the relative kinematics vector equations as:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{y}}_{\mathbf{ji}} = \underline{\mathbf{v}}_{\mathbf{ji}} \tag{5.7}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{v}}_{\mathbf{ji}} = \underline{\mathbf{c}}_{\mathbf{j}} - \underline{\mathbf{c}}_{\mathbf{i}} \tag{5.8}$$

Where:

$\underline{\mathbf{y}}_{\mathbf{i}} = [\mathbf{x_i} \quad \mathbf{z_i}]^{\mathbf{T}}$ **:** is the position vector of vehicle **i** in fixed axis.

$\underline{\mathbf{v}}_{\mathbf{i}} = [\mathbf{u_i} \quad \mathbf{w_i}]^{\mathbf{T}}$ **:** is the velocity vector of vehicle **i** in fixed axis.

$\underline{\mathbf{c}}_{\mathbf{i}} = [\mathbf{a_{x_i}} \quad \mathbf{a_{z_i}}]^{\mathbf{T}}$ **:** is the target acceleration vector of vehicle **i** in fixed axis.

$\underline{\mathbf{y}}_{\mathbf{ji}} = [\mathbf{x_{ji}} \quad \mathbf{z_{ji}}]^{\mathbf{T}}$ **:** position vector of vehicle **j** w.r.t **i** in fixed axis.

$\underline{\mathbf{v}}_{\mathbf{ji}} = [\mathbf{u_{ji}} \quad \mathbf{w_{ji}}]^{\mathbf{T}}$ **:** velocity vector of vehicle **j** w.r.t **i** in fixed axis.

$\underline{\mathbf{c}}_{\mathbf{ji}} = [\mathbf{a_{x_{ji}}} \quad \mathbf{a_{z_{ji}}}]^{\mathbf{T}} = \underline{\mathbf{c}}_{\mathbf{j}} - \underline{\mathbf{c}}_{\mathbf{i}}$ **:** acceleration vector of vehicle **j** w.r.t **i** in fixed axis.

5.2 Constructing Relative Sightline (LOS) Angles and Rates - (Rotational Kinematics)

The separation range $\mathbf{R_{ji}}$ between vehicles **j** w.r.t **i** is given by:

$$\mathbf{R_{ji}} = \left(\mathbf{x_{ji}}^{\mathbf{2}} + \mathbf{z_{ji}}^{\mathbf{2}}\right)^{\frac{1}{2}} = \left(\underline{\mathbf{y}}_{\mathbf{ji}}^{\mathbf{T}}\underline{\mathbf{y}}_{\mathbf{ji}}\right)^{\frac{1}{2}} \tag{5.9}$$

The range rate $\dot{\mathbf{R}}_{\mathbf{ji}}$ is given by:

$$\frac{d}{dt}\mathbf{R}_{ji} = \dot{\mathbf{R}}_{ji} = \frac{\mathbf{x}_{ji}\mathbf{u}_{ji} + \mathbf{z}_{ji}\mathbf{w}_{ji}}{\mathbf{R}_{ji}} = \frac{\left(\underline{\mathbf{y}}_{ji}{}^{\mathbf{T}}\underline{\mathbf{v}}_{ji}\right)}{\mathbf{R}_{ji}} \tag{5.10}$$

The Closing Velocity $\mathbf{V}_{\mathbf{cji}}$ is given by:

$$\mathbf{V}_{\mathbf{c}_{\mathbf{ji}}} = -\dot{\mathbf{R}}_{\mathbf{ji}} \tag{5.11}$$

The sightline angle $\gamma_{\mathbf{ji}}$ of j w.r.t i is given by:

$$\tan\gamma_{\mathbf{ji}} = \frac{\mathbf{z}_{\mathbf{ji}}}{\mathbf{x}_{\mathbf{ji}}} \tag{5.12}$$

Differentiating both sides and simplifying, gives us:

$$\frac{d}{dt}\gamma_{\mathbf{ji}} = \dot{\gamma}_{\mathbf{ji}} = \frac{\dot{\mathbf{z}}_{\mathbf{ji}}\mathbf{x}_{\mathbf{ji}}}{\mathbf{R}_{\mathbf{ji}}{}^{2}} - \frac{\mathbf{z}_{\mathbf{ji}}\dot{\mathbf{x}}_{\mathbf{ji}}}{\mathbf{R}_{\mathbf{ji}}{}^{2}} = \frac{\mathbf{x}_{\mathbf{ji}}\mathbf{w}_{\mathbf{ji}} - \mathbf{z}_{\mathbf{ji}}\mathbf{u}_{\mathbf{ji}}}{\mathbf{R}_{\mathbf{ji}}{}^{2}} = \frac{\underline{\mathbf{y}}_{\mathbf{ji}}{}^{\mathbf{T}}[\mathbf{J}]\underline{\mathbf{v}}_{\mathbf{ji}}}{\mathbf{R}_{\mathbf{ji}}{}^{2}} \tag{5.13}$$

Note that an approximation to (5.13) is sometimes used, although not recommended for simulation purposes, based on the assumptions that the engagement geometry does not deviate significantly from the collision course; in this case:
$\mathbf{x}_{\mathbf{ji}} \approx \mathbf{R}_{\mathbf{ji}} = \mathbf{V}_{\mathbf{c}_{\mathbf{ji}}}\mathbf{T}_{\mathbf{go}}$; $\dot{\mathbf{x}}_{\mathbf{ji}} \approx \dot{\mathbf{R}}_{\mathbf{ji}} = -\mathbf{V}_{\mathbf{c}_{\mathbf{ji}}}$; $\mathbf{T}_{\mathbf{go}} = (\mathbf{t}_{\mathbf{f}} - \mathbf{t})$**:** is the time-to-go.

Substituting this in equation (5.13) gives us:

$$\dot{\gamma}_{\mathbf{ji}} = \frac{1}{\mathbf{V}_{\mathbf{c}_{\mathbf{ji}}}}\left(\frac{\dot{\mathbf{z}}_{\mathbf{ji}}}{\mathbf{T}_{\mathbf{go}}} + \frac{\mathbf{z}_{\mathbf{ji}}}{\mathbf{T}_{\mathbf{go}}{}^{2}}\right) \tag{5.14}$$

The measurements $\hat{\dot{\gamma}}_{\mathbf{ji}}$ obtained from the seeker that are used to construct the guidance commands are given by:

$$\hat{\dot{\gamma}}_{\mathbf{ji}} = \dot{\gamma}_{\mathbf{ji}} + \Delta\dot{\gamma}_{\mathbf{ji}} \tag{5.15}$$

Where:
$\Delta\dot{\gamma}_{\mathbf{ji}}$ **:** seeker LOS rate measurement error.

The above relationships (5.9)-(5.15) will also be referred to as the seeker model.

5.3 Vehicle Navigation Model

Let us define the following:

$\mathbf{a}^{\mathbf{b}}_{\mathbf{x}_{\mathbf{i}}}$ **:** **x**-acceleration achieved by vehicle **i** in its body axis.

$\mathbf{a}^{\mathbf{b}}_{\mathbf{z}_{\mathbf{i}}}$ **:** **y**-acceleration achieved by vehicle **i** in its body axis.

The transformation from fixed to body axis is given by (see Figure 2):

$$\begin{bmatrix} a^b_{x_i} \\ a^b_{z_i} \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \begin{bmatrix} a_{x_i} \\ a_{y_i} \end{bmatrix} = \left[T^b_f\right]_i \begin{bmatrix} a_{x_i} \\ a_{z_i} \end{bmatrix} \tag{5.16}$$

In vector/matrix notation this equation may be written as:

$$\underline{c}^b_i = \left[T^b_f\right]_i \underline{c}_i \tag{5.17}$$

$$\underline{c}_i = \left[T^f_b\right]_i \underline{c}^b_i \tag{5.18}$$

Where:

ψ_i : body (Euler) angle of the **i**th vehicle w.r.t the fixed axis.

$\underline{c}^b_i = \begin{bmatrix} a^b_{x_i} & a^b_{y_i} \end{bmatrix}^T$ **:** is acceleration vector of vehicle **i** in its body axis.

$\left[T^f_b\right]_i = \left[T^b_f\right]^T_i = \left[T^b_f\right]^{-1}_i$ **:** is the transformation matrix from body axis to fixed axis.

$\left[T^f_b\right]_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$ **:** is the (direction cosine) transformation matrix.

The vehicle velocity is given by:

$$V_i = \left(u_i^2 + w_i^2\right)^{\frac{1}{2}} = \left(\underline{v}_i^T \underline{v}_i\right)^{\frac{1}{2}} \tag{5.19}$$

The body flight path angle is given by:

$$\tan(\theta_i - \alpha_i) = \frac{w_i}{u_i}$$

Where: α_i **:** is the body angle of incidence angle.

Differentiating both sides and simplifying and assuming (as in section 2.3) that $\alpha_i, \dot{\alpha}_i$ are small, we get:

$$\frac{d}{dt}\theta_i = \dot{\theta}_i = \frac{\dot{w}_i u_i}{V_i^2} - \frac{w_i \dot{u}_i}{V_i^2} = \frac{u_i a_{z_i} - w_i a_{x_i}}{V_i^2} = \frac{\underline{v}_i^T [J] \underline{c}_i}{V_i^2} \tag{5.20}$$

5.4 Vehicle Autopilot Dynamics

Assuming a first order lag for the autopilot, we may write for vehicle i:

$$\frac{d}{dt} a^b_{x_i} = -\tau_{x_i} a^b_{x_i} + \tau_{x_i} a^b_{x_{i_d}} \tag{5.21}$$

$$\frac{d}{dt} a^b_{z_i} = -\tau_{z_i} a^b_{z_i} + \tau_{z_i} a^b_{z_{i_d}} \tag{5.22}$$

In vector/matrix notation equations (5.21), (5.22) may be written as:

$$\frac{d}{dt}\underline{c}_i^b = [-\Lambda_i]\underline{c}_i^b + [\Lambda_i]\underline{c}_{i_d}^b \quad (5.23)$$

Where:
τ_{x_i} : Vehicle i autopilot's longitudinal time-constant.
τ_{z_i} : Vehicle i autopilot's lateral time-constant.
$[\Lambda_i] = \begin{bmatrix} \tau_{x_i} & 0 \\ 0 & \tau_{z_i} \end{bmatrix}$
$a_{x_{i_d}}^b$: x-acceleration demanded by vehicle i in its body axis.
$a_{z_{i_d}}^b$: y-acceleration demanded by vehicle i in its body axis.
$\underline{c}_{i_d}^b = \begin{bmatrix} a_{x_{i_d}}^b & a_{z_{i_d}}^b \end{bmatrix}^T$: is the demanded missile acceleration (command input) vector in body axis.

Remarks: As in the case of azimuth plane engagement, the longitudinal guidance commands $a_{x_{i_d}}^b = \frac{\delta T_i - \delta D_i}{m_i}$ are assumed to be zero. The effects of aerodynamic forces under nominal flight conditions i.e. $\bar{a}_{z_i}^b = \frac{\bar{Z}}{m} + g\cos\theta$ and $\bar{a}_{x_i}^b = \frac{(\bar{T} - \bar{D})}{m} - g\sin\theta$ (see Appendix A) have to be included in the simulation; these are shown as: $\underline{\bar{c}}_i^b$ in the block diagram Figure 7. The limits on the lateral acceleration demanded can be implemented as described in Appendix A, i.e. $\left\| a^b{}_{z_{i_d}} \right\| \le \mu\, a_{z_{max}}$.

5.5 Guidance Laws-PN and APN Guidance

Several versions of the guidance laws derived in section 3 for the azimuth plane engagement and these extend directly to the elevation plane case. These are briefly discussed below:

5.5.1. Version 1 (PN-1):

$$\dot{\theta}_{i_d} = N\dot{\gamma}_{ji} \quad (5.25)$$

Where: N : is the navigation constant; and the demanded attacker lateral acceleration is given by:

$$a_{z_{i_d}}^b = V_i\dot{\theta}_{i_d} = NV_i\dot{\gamma}_{ji} \quad (5.26)$$

$$a_{x_{i_d}}^b = \frac{\delta T_i - \delta D_i}{m_i} \quad (5.27)$$

5.5.2. Version 2 (PN-2):

$$a_{n_{ji}} = V_{c_{ji}}\dot{\gamma}_{ji} \quad (5.28)$$

$$\mathbf{a^b_{z_{i_d}}} = \mathbf{N a_{n_{ji}}} = \mathbf{N V_{c_{ji}}} \, \dot{\gamma}_{\mathbf{ji}} \quad (5.29)$$

$$\mathbf{a^b_{x_{i_d}}} = \frac{\delta \mathbf{T_i} - \delta \mathbf{D_i}}{\mathbf{m_i}} \quad (5.30)$$

Where:
$\mathbf{V_{c_{ji}}} = -\dot{\mathbf{R}}_{\mathbf{ji}}$ **:** is the '*closing velocity*' between the attacker and the target.
$\mathbf{a_{n_{ji}}}$ **:** is the acceleration normal to the LOS.

5.5.3. Version 3 (PN-3):

$$\mathbf{a^b_{z_{i_d}}} = \mathbf{N a_{n_{ji}}} \cos\left(\theta_{\mathbf{i}} - \gamma_{\mathbf{ji}}\right) = \mathbf{N V_{c_{ji}}} \cos\left(\theta_{\mathbf{i}} - \gamma_{\mathbf{ji}}\right)\dot{\gamma}_{\mathbf{ji}} \quad (5.31)$$

$$\mathbf{a^b_{x_{i_d}}} = \frac{\delta \mathbf{T_i} - \delta \mathbf{D_i}}{\mathbf{m_i}} \quad (5.32)$$

5.5.4. Augmented Proportional Navigation (APN) Guidance

Finally, a variation of the PN guidance law is the APN that can be implemented as follows:

$$\underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i_d}} = (\mathbf{PNG}) + \mathbf{N'}\left[\mathbf{T^b_f}\right]_{\mathbf{i}} \underline{\mathbf{c}}_{\mathbf{j}} \quad (5.33)$$

Where:
$\mathbf{N'}$ **:** is the (target) acceleration navigation constant
$(\mathbf{PNG})$ **:** is the proportional navigation guidance law given in (3.1)-(3.7)

5.6 Overall State Space Model in Elevation Plane

The overall non-linear state space model (e.g. for APN guidance) that can be used for sensitivity studies and for non-linear or Monte-Carlo analysis is given below:

$$\frac{\mathbf{d}}{\mathbf{dt}} \underline{\mathbf{y}}_{\mathbf{ji}} = \underline{\mathbf{v}}_{\mathbf{ji}} \quad (6.1)$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \underline{\mathbf{v}}_{\mathbf{ji}} = \left[\mathbf{T^f_b}\right]_{\mathbf{j}} \underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{j}} - \left[\mathbf{T^f_b}\right]_{\mathbf{i}} \underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i}} \quad (6.2)$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \gamma_{\mathbf{ji}} = \frac{\underline{\mathbf{y}}_{\mathbf{ji}}{}^{\mathbf{T}} [\mathbf{J}] \underline{\mathbf{v}}_{\mathbf{ji}}}{\underline{\mathbf{y}}_{\mathbf{ji}}{}^{\mathbf{T}} \underline{\mathbf{y}}_{\mathbf{ji}}} \quad (6.3)$$

$$\underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i_d}} = \left\{(\mathbf{PNG}) + \mathbf{N'}\left[\mathbf{T^b_f}\right]_{\mathbf{i}} \underline{\mathbf{c}}_{\mathbf{j}}\right\} \quad (6.4)$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i}} = \left[-\Lambda_{\mathbf{i}}\right] \underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i}} + \left[\Lambda_{\mathbf{i}}\right] \underline{\mathbf{c}}^{\mathbf{b}}_{\mathbf{i_d}} \quad (6.5)$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \theta_{\mathbf{i}} = \dot{\theta}_{\mathbf{i}} = \frac{\underline{\mathbf{v}}_{\mathbf{i}}{}^{\mathbf{T}} [\mathbf{J}] \underline{\mathbf{c}}_{\mathbf{i}}}{\underline{\mathbf{v}}_{\mathbf{i}}{}^{\mathbf{T}} \underline{\mathbf{v}}_{\mathbf{i}}} \quad (6.6)$$

The overall model that can be implemented on the computer is given in Table 5.1. and the block-diagram is shown in Figure 7.

6. Conclusions

In this report mathematical models are derived for multi-vehicle guidance, navigation and control model suitable for developing, implementing and testing modern missile guidance systems. The models allow for incorporating changes in body attitude in addition to autopilot lags, accelerometer limits and aerodynamic effects. These models will be found to be particularly suitable for studying the performance of both the conventional and the modern guidance such as those that arise of game theory and intelligent control theory. The following are considered to be the main contribution of this report:

- The models are derived for Az as well as the El plane engagements for multi-vehicle engagement scenarios,
- The models incorporate non-linear effects including large changes in vehicle body attitude, autopilot lags, acceleration limits and aerodynamic effects,
- The models presented in this report can easily be adapted for multi-run non-linear analysis of guidance performance and for undertaking Monte-Carlo analysis.

7. References

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Table 1: Combined (Yaw-Plane) State Space Dynamics Model for Navigation, Seeker, Guidance and Autopilot

	ALGORITHM	MODULE NAME
1	$\frac{d}{dt}\underline{x}_i = \underline{u}_i$ $\frac{d}{dt}\underline{u}_i = \underline{a}_i$	Kinematics
2	$R_{ji} = \left(\underline{x}_{ji}{}^T \underline{x}_{ji}\right)^{\frac{1}{2}}$ $\frac{d}{dt}R_{ji} = \dot{R}_{ji} = \frac{\left(\underline{x}_{ji}{}^T \underline{u}_{ji}\right)}{R_{ji}}$ $V_{c_{ji}} = -\dot{R}_{ji}$ $\frac{d}{dt}\lambda_{ji} = \frac{\underline{x}_{ji}{}^T [J] \underline{u}_{ji}}{R_{ji}{}^2}$ $\dot{\hat{\lambda}}_{ji} = \dot{\lambda}_{ji} + \Delta\dot{\lambda}_{ji}$	Seeker
3	1. $a^b_{y_d} = NV_i \dot{\hat{\lambda}}_{ji}$ 2. $a^b_{y_{i_d}} = NV_{c_{ji}} \dot{\lambda}_{ji}$ 3. $a^b_{y_{i_d}} = NV_{c_{ji}} \cos\left(\psi_i - \lambda_{ji}\right) \dot{\lambda}_{ji}$ 4. $\underline{a}^b_{i_d} = (PNG) + N' \left[T^b_f\right]_i \underline{a}_j$	Guidance
4	$\frac{d}{dt}\underline{a}^b_i = [-\Lambda_i]\underline{a}^b_i + [\Lambda_i]\underline{a}^b_{i_d}$	Autopilot
5	$V_i = \left(\underline{u}_i{}^T \underline{u}_i\right)^{\frac{1}{2}}$ $\frac{d}{dt}\psi_i = \frac{\underline{u}_i{}^T [J]\, \underline{a}_i}{V_i{}^2}$ $\left[T^f_b\right]_i = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}$; $\left[T^f_b\right]_i = \left[T^b_f\right]^T_i$ $\underline{a}_i = \left[T^f_b\right]_i \underline{a}^b_i$	Navigation

Table 2: Combined (Pitch-Plane) State Space Dynamics Model for Navigation, Seeker, Guidance and Autopilot

	ALGORITHM	MODULE NAME
1	$\frac{d}{dt}\underline{y}_i = \underline{v}_i$ $\frac{d}{dt}\underline{v}_i = \underline{c}_i$	Kinematics
2	$R_{ji} = \left(\underline{y}_{ji}{}^T \underline{y}_{ji}\right)^{\frac{1}{2}}$ (2.6) $\frac{d}{dt}R_{ji} = \dot{R}_{ji} = \frac{\left(\underline{y}_{ji}{}^T \underline{v}_{ji}\right)}{R_{ji}}$ $V_{c_{ji}} = -\dot{R}_{ji}$ $\frac{d}{dt}\gamma_{ji} = \frac{\underline{y}_{ji}{}^T [J] \underline{v}_{ji}}{R_{ji}{}^2}$ $\dot{\hat{\gamma}}_{ji} = \dot{\gamma}_{ji} + \Delta\dot{\gamma}_{ji}$	Seeker
3	1. $a^b_{z_d} = NV_i \dot{\gamma}_{ji}$ 2. $a^b_{z_{i_d}} = NV_{c_{ji}} \dot{\gamma}_{ji}$ 3. $a^b_{z_{i_d}} = NV_{c_{ji}} \cos(\theta_i - \gamma_{ji})\, \dot{\gamma}_{ji}$ 4. $\underline{c}^b_{i_d} = (PNG) + N' \left[T^b_f\right]_i \underline{c}_j$;	Guidance
4	$\frac{d}{dt}\underline{c}^b_i = [-\Lambda_i]\underline{c}^b_i + [\Lambda_i]\underline{c}^b_{i_d}$	Autopilot
5	$V_i = \left(\underline{v}_i{}^T \underline{v}_i\right)^{\frac{1}{2}}$ $\frac{d}{dt}\theta_i = \frac{\underline{v}_i{}^T [J] \underline{c}_i}{V_i{}^2}$ $\left[T^f_b\right]_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$; $\left[T^f_b\right]_i = \left[T^b_f\right]^T_i$ $\underline{c}_i = \left[T^f_b\right]_i \underline{c}^b_i$	Navigation

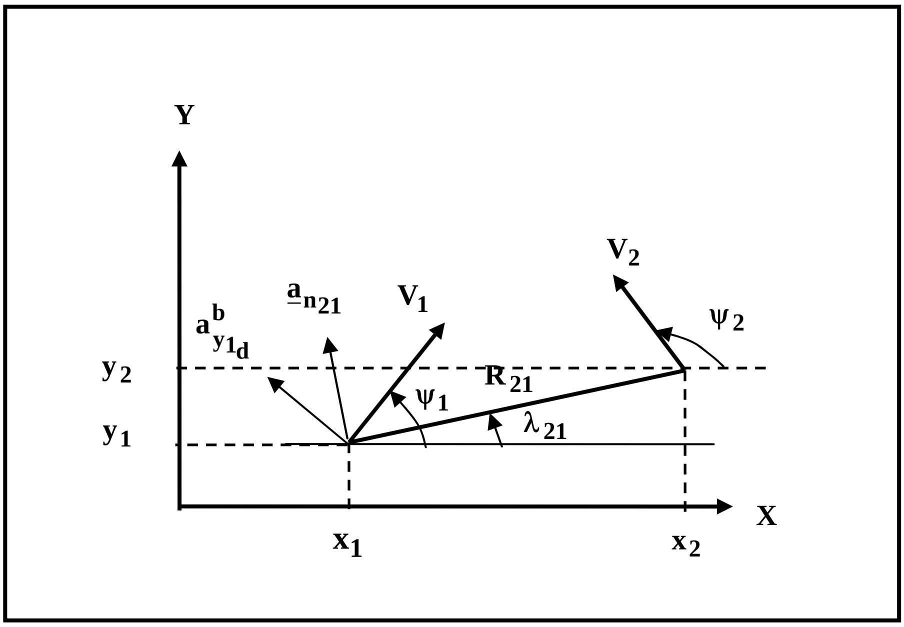

Figure 1: Engagement Geometry for 2-Vehicles

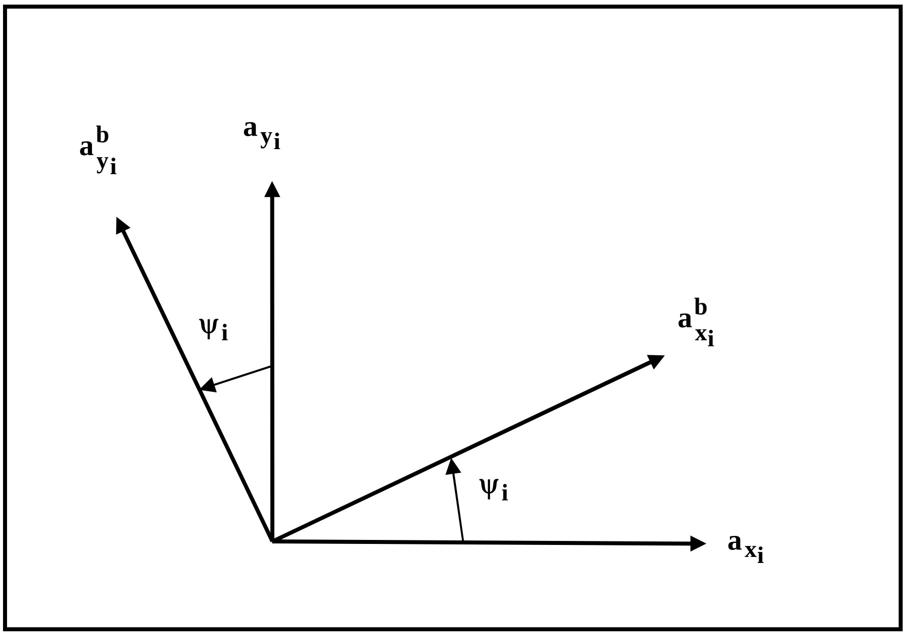

Figure 2: Axis transformation fixed <=>body in pitch-plane

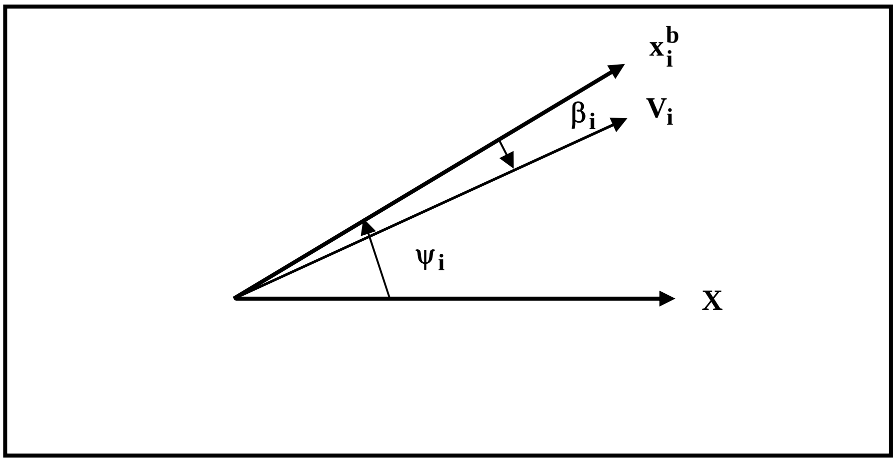

Figure 3: Body Incidence

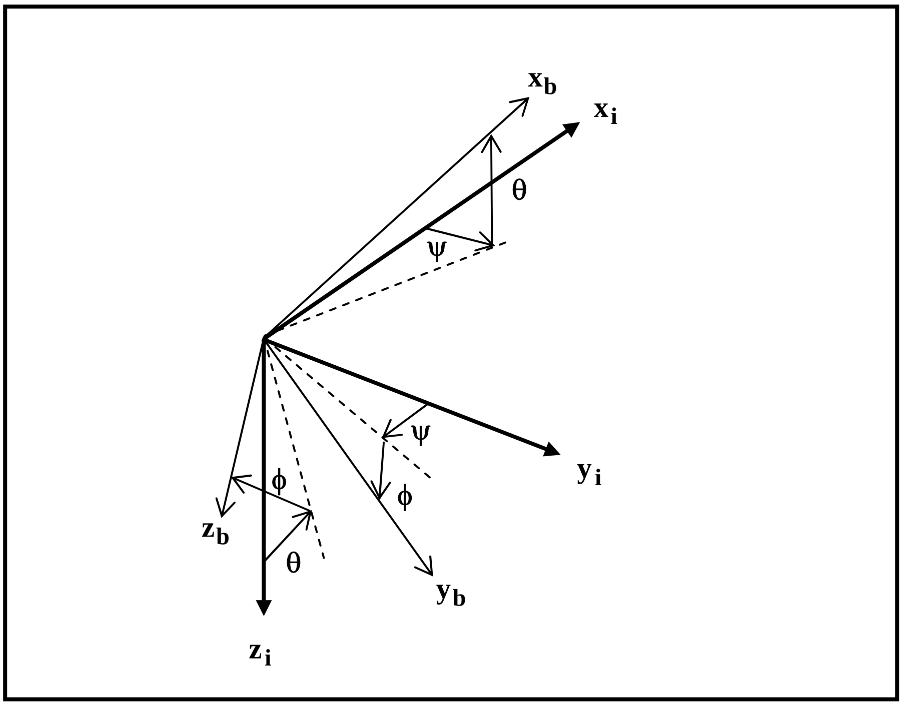

Figure 4: Axis System Convention

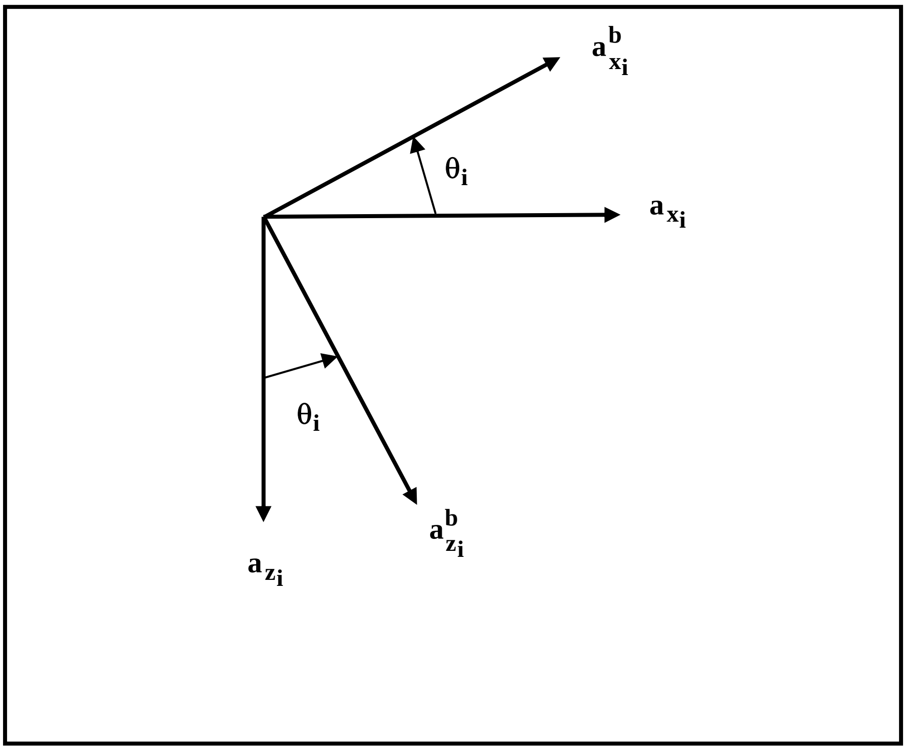

Figure 5: Axis transformation fixed <=>body in pitch-plane

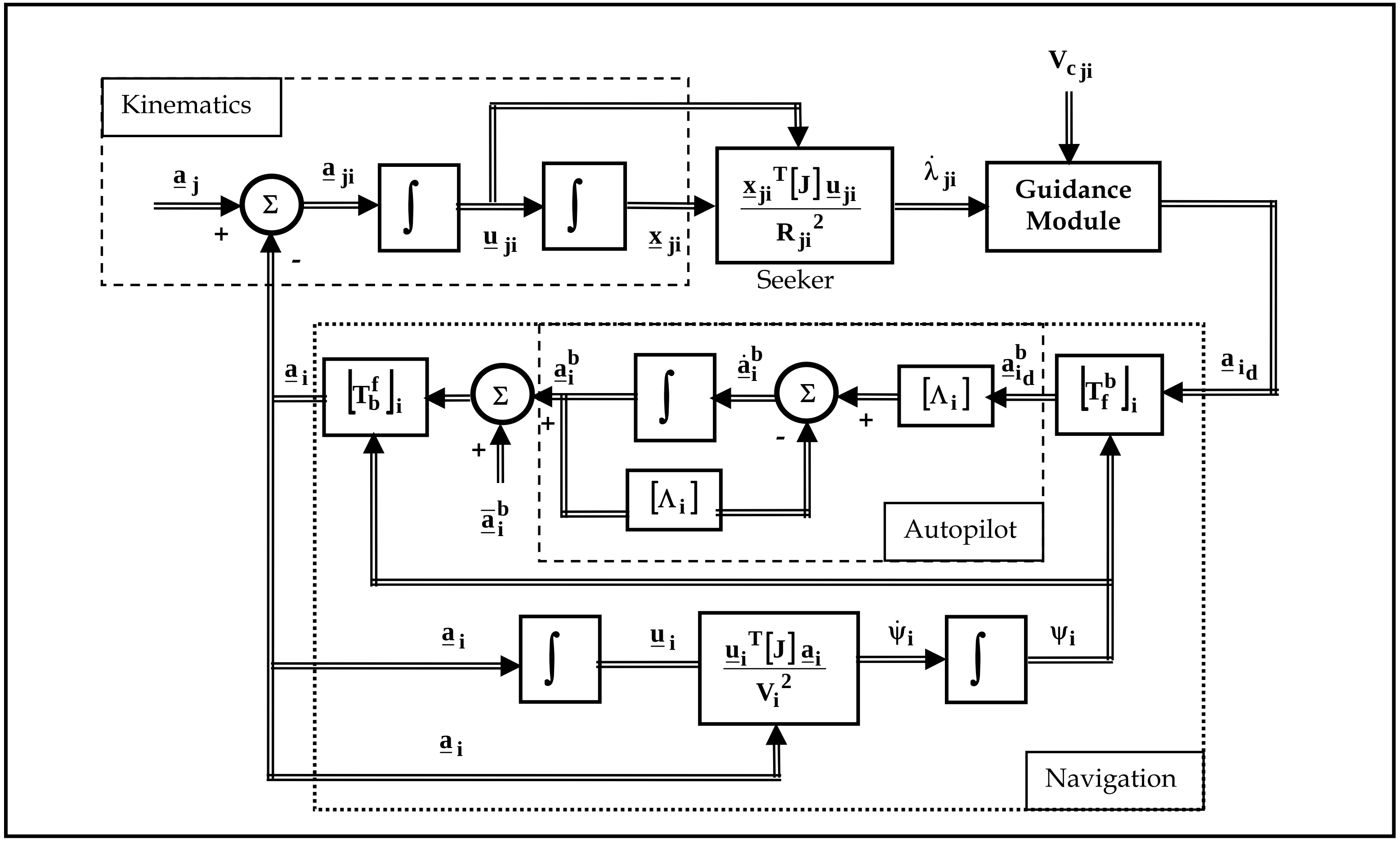

Figure 6: Yaw-Plane Simulation Model Block Diagram

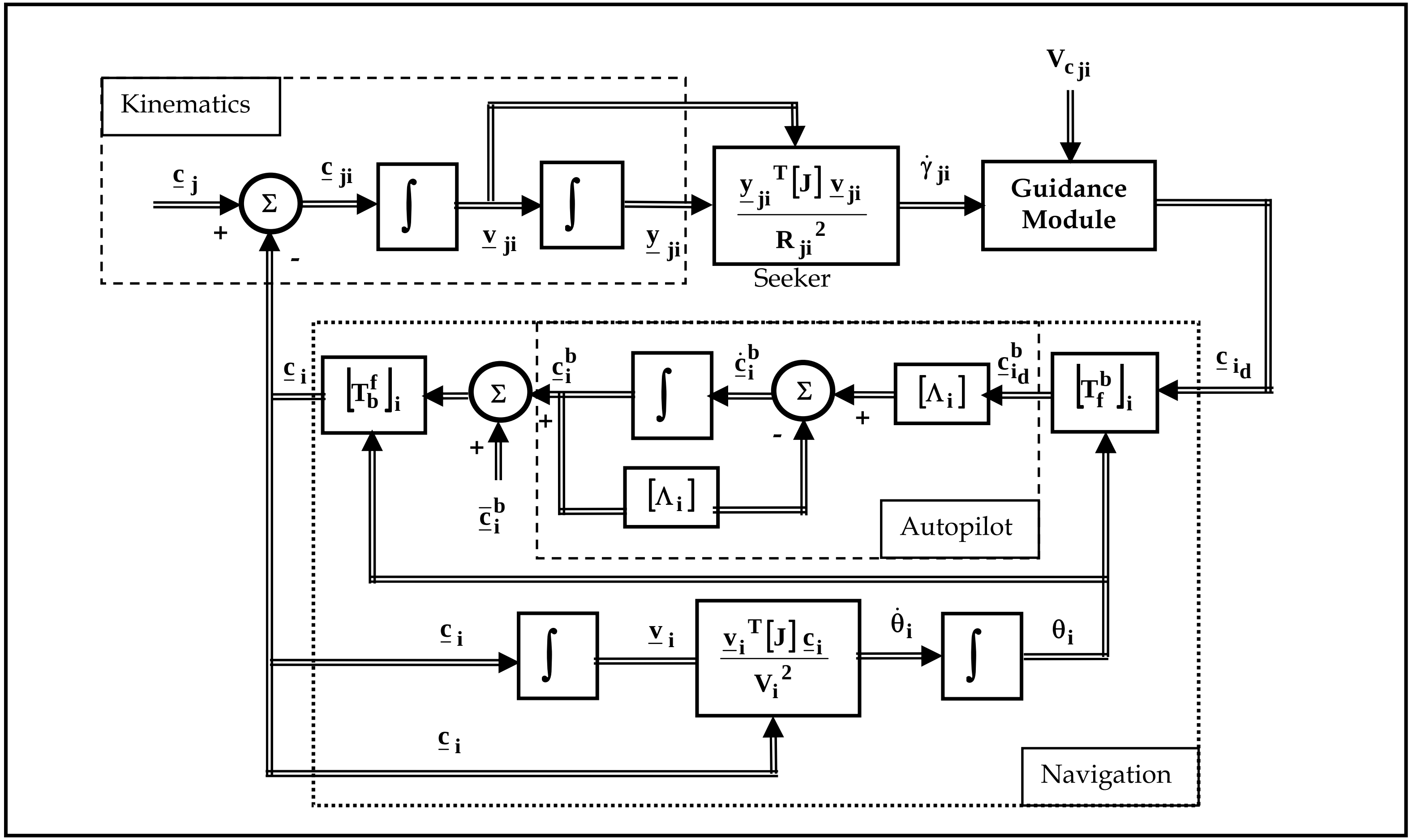

Figure 7: Pitch-Plane Simulation Model Block Diagram

Appendix A

A.1 Aerodynamic Forces and Equations of Motion

For a symmetrical body $\left(\mathbf{I_{zx}=0; I_y=I_z}\right)$, the equations of motion for an aerodynamic vehicle are given by (see Figure A1.1) [4]:

$$\mathbf{\dot{u}^b + qw^b - rv^b = \frac{X}{m} - g\sin\theta} \quad \text{(A-1.1)}$$

$$\mathbf{\dot{v}^b + ru^b - pw^b = \frac{Y}{m} + g\cos\theta\sin\phi} \quad \text{(A-1.2)}$$

$$\mathbf{\dot{w}^b + pv^b - qu^b = \frac{Z}{m} + g\cos\theta\cos\phi} \quad \text{(A-1.3)}$$

$$\mathbf{\dot{p} + qr\frac{\left(I_z - I_y\right)}{I_x} = \frac{L}{I_x}} \quad \text{(A-1.4)}$$

$$\mathbf{\dot{q} + rp\frac{\left(I_x - I_z\right)}{I_y} = \frac{M}{I_y}} \quad \text{(A-1.5)}$$

$$\mathbf{\dot{r} + pq\frac{\left(I_y - I_x\right)}{I_z} = \frac{N}{I_z}} \quad \text{(A-1.6)}$$

Where:
$\left(\mathbf{u^b, v^b, w^b}\right)$**:** are the vehicle velocities defined in body axis.
$\left(\mathbf{p, q, r}\right)$**:** are the body rotation rates w.r.t the fixed axis defined in the body axis.
$\left(\mathbf{X, Y, Z}\right)$**:** are the aerodynamic forces acting on the vehicle body defined in the body axis.
$\left(\mathbf{L, M, N}\right)$**:** are the aerodynamic moments acting on the vehicle body defined in the body axis.
$\left(\mathbf{I_x, I_y, I_z}\right)$**:** are the vehicle body inertias.
$\mathbf{m}$ **:** is the vehicle mass.
$\left(\psi, \theta, \phi\right)$**:** are respectively the body (Euler) angles w.r.t the fixed axis.

For a non rolling vehicle $\mathbf{\dot{p} = p = \phi = 0}$; this assumption enables us to decouple the yaw and pitch kinematics. Equations (A-1.1)-(A1.6) give us:

$$\mathbf{\dot{u}^b + qw^b - rv^b = \frac{X}{m} - g\sin\theta} \quad \text{(A-1.7)}$$

$$\mathbf{\dot{v}^b + ru^b = \frac{Y}{m}} \quad \text{(A-1.8)}$$

$$\mathbf{\dot{w}^b - qu^b = \frac{Z}{m} + g\cos\theta} \quad \text{(A-1.9)}$$

$$\mathbf{L = 0} \quad \text{(A-1.10)}$$

$$\mathbf{\dot{q} = \frac{M}{I_y}} \quad \text{(A-1.11)}$$

$$\mathbf{\dot{r} = \frac{N}{I_z}} \quad \text{(A-1.12)}$$

The accelerations about the vehicle body CG is given by:

$$\mathbf{a_x^b} = \mathbf{\dot{u}^b} + \mathbf{qw^b} - \mathbf{rv^b} = \frac{\mathbf{X}}{\mathbf{m}} - \mathbf{g}\sin\theta \quad \text{(A-1.13)}$$

$$\mathbf{a_y^b} = \mathbf{\dot{v}^b} + \mathbf{ru^b} = \frac{\mathbf{Y}}{\mathbf{m}} \quad \text{(A-1.14)}$$

$$\mathbf{a_z^b} = \mathbf{\dot{w}^b} - \mathbf{qu^b} = \frac{\mathbf{Z}}{\mathbf{m}} + \mathbf{g}\cos\theta \quad \text{(A-1.15)}$$

Where: $\left(\mathbf{a_x^b}, \mathbf{a_y^b}, \mathbf{a_z^b}\right)$**:** are the body accelerations w.r.t the fixed axis defined in the body axis.

If we consider perturbation about the nominal, we get:

$$\mathbf{\bar{a}_x^b} + \delta\mathbf{a_x^b} = \frac{\left(\mathbf{\bar{X}} + \delta\mathbf{X}\right)}{\mathbf{m}} - \mathbf{g}\left(\sin\bar{\theta} + \cos\bar{\theta}\,\delta\theta\right)$$

$$\mathbf{\bar{a}_y^b} + \delta\,\mathbf{a_y^b} = \frac{\left(\mathbf{\bar{Y}} + \delta\mathbf{Y}\right)}{\mathbf{m}}$$

$$\mathbf{\bar{a}_z^b} + \delta\,\mathbf{a_z^b} = \frac{\left(\mathbf{\bar{Z}} + \delta\mathbf{Z}\right)}{\mathbf{m}} + \mathbf{g}\left(\cos\bar{\theta} - \sin\bar{\theta}\,\delta\theta\right)$$

A.2 2-D Yaw-Plane Kinematics Equations:

For 2-D yaw-plane kinematics only $\bar{\theta} = 0; \delta\theta = 0$ (i.e. zero pitch motion), therefore, the X and Y-plane *steady state* equations (in body axis) may be written as:

$$\mathbf{\bar{a}_y^b} = \frac{\mathbf{\bar{Y}}}{\mathbf{m}} = \mathbf{\tilde{Y}} \quad \text{(A-1.16)}$$

$$\mathbf{\bar{a}_x^b} = \frac{\mathbf{\bar{X}}}{\mathbf{m}} = \frac{\left(\mathbf{\bar{T}} - \mathbf{\bar{D}}\right)}{\mathbf{m}} = \left(\mathbf{\tilde{T}} - \mathbf{\tilde{D}}\right) \quad \text{(A-1.17)}$$

Where we define: $\frac{\mathbf{\bar{Y}}}{\mathbf{m}} = \mathbf{\tilde{Y}}$; $\frac{\mathbf{\bar{X}}}{\mathbf{m}} = \frac{\left(\mathbf{\bar{T}} - \mathbf{\bar{D}}\right)}{\mathbf{m}} = \left(\mathbf{\tilde{T}} - \mathbf{\tilde{D}}\right)$. Also, the total thrust is defined as: $\mathbf{T} = \mathbf{\bar{T}} + \delta\mathbf{T}$, and the total drag is defined as: $\mathbf{D} = \mathbf{\bar{D}} + \delta\mathbf{D}$.

For 'nominal flight' condition in the yaw-plane $\mathbf{\bar{Y}} = 0$; and the perturbation equation is given by:

$$\delta\,\mathbf{a_y^b} = \frac{\delta\mathbf{Y}}{\mathbf{m}} = \delta\mathbf{\tilde{Y}} \quad \text{(A-1.18)}$$

$$\delta\mathbf{a_x^b} = \frac{\delta\mathbf{X}}{\mathbf{m}} = \frac{\left(\delta\mathbf{T} - \delta\mathbf{D}\right)}{\mathbf{m}} = \left(\delta\mathbf{\tilde{T}} - \delta\mathbf{\tilde{D}}\right) \quad \text{(A-1.19)}$$

Where :

$\delta\,\mathbf{a_y^b}$ **:** represents the body axis guidance commands (lateral acceleration) applied by the vehicle.

δa_x^b **:** represents the body axis guidance commands (longitudinal acceleration) applied by the vehicle. However, during guidance manoeuvre $(\delta\tilde{T}, \delta\tilde{D})$ are not directly controlled, Hence we may assume δa_x^b to be zero.

A.3 2-D Pitch-Plane Kinematics Equations:

Unlike the previous case, for 2-D pitch-plane kinematics $\bar{\theta} \neq 0$, and for steady state conditions, we get:

$$\bar{a}_z^b = \frac{\bar{Z}}{m} + g\cos\bar{\theta} = \tilde{Z} + g\cos\bar{\theta} \tag{A-1.20}$$

$$\bar{a}_x^b = \frac{\bar{X}}{m} - g\sin\bar{\theta} = \frac{(\bar{T} - \bar{D})}{m} - g\sin\bar{\theta} = (\tilde{T} - \tilde{D}) - g\sin\bar{\theta} \tag{A-1.21}$$

The X , Z (pitch)-plane perturbation kinematics (in body axis) is given by:

$$\delta a_z^b = \frac{\delta Z}{m} = \delta\tilde{Z} \tag{A-1.22}$$

$$\delta a_x^b = \frac{(\delta T - \delta D)}{m} = (\delta\tilde{T} - \delta\tilde{D}) \tag{A-1.23}$$

Where

δa_z^b **:** represents the body axis guidance commands (lateral acceleration) applied by the vehicle.

As in the case of the yaw-plane, during guidance $(\delta\tilde{T}, \delta\tilde{D})$ are not directly controlled, hence we may assume δa_x^b to be zero. The reader will recognize, that in the main text of this report:

$$a_{x_{d_i}}^b \equiv \delta a_x,\; a_{y_{d_i}}^b \equiv \delta a_y \;\; a_{z_{d_i}}^b \equiv \delta a_z \tag{A-1.24}$$

A.4 Calculating the Aerodynamic Forces

For the purposes of the simulation under consideration we may assume that the vehicle thrust profile $\bar{T}(t)$, say as a function of time, is given; then the drag force $\bar{D}$, which depends on the vehicle aerodynamic configuration, is given by:

$$\bar{D} = \left(\frac{1}{2}\rho\bar{V}^2 S\right) C_D(\bar{\alpha}, \bar{\beta}) \tag{A1.25}$$

$$\bar{Y} = \left(\frac{1}{2}\rho\bar{V}^2 S\right) C_L(\bar{\beta}) = 0 \tag{A1.26}$$

$$\bar{Z} = \left(\frac{1}{2}\rho\bar{V}^2 S\right) C_L(\bar{\alpha}) = -g\cos\bar{\theta} \tag{A1.27}$$

Where: the term in the bracket is the dynamic pressure; ρ being the air density, S is the body characteristic surface area and $\bar{V}$ is the steady-state velocity. C_D is the drag coefficient and

C_L is the lift coefficient. $(\bar{\alpha},\bar{\beta})$ represent respectively the pitch- and the yaw-plane nominal (steady-state) incidence angles. Contribution to thrust and drag due to control deflections are small and ignored. Also:

$$\delta Y = \left(\frac{1}{2}\rho V^2 S\right) C_L(\delta\beta) = 0 \qquad (A1.28)$$

$$\delta Z = \left(\frac{1}{2}\rho V^2 S\right) C_L(\delta\alpha) = -g \sin\bar{\theta}\,\delta\theta \qquad (A1.29)$$

$(\delta\alpha,\delta\beta)$ represent respectively the the variation in pitch- and the yaw-plane incidence angles as a result of control demands; these are assumed to be small.

Note that for a given $(\delta\alpha,\delta\beta)$, $\delta Y,\delta Z \propto V^2$, the maximum/minimum acceleration capability of a vehicle is rated at the nominal velocity $\bar{V}$, then the maximum/minimum acceleration at any other velocity V is given by:

$$\left\| a^b_{y_d} \right\| \le \mu\, a^b_{y_{max}}\ ;\ \left\| a^b_{z_d} \right\| \le \mu\, a^b_{z_{max}}\quad ;\text{Where: } \mu = \left(\frac{V}{\bar{V}}\right)^2$$

A.5 Body Incidence

The body incidence angles (α,β) are given by $(v_b, w_b << u_b)$:

$\alpha = \tan^{-1}\left(\frac{w^b}{u^b}\right) \approx \frac{w^b}{u^b}$; $\beta = \tan^{-1}\left(\frac{v^b}{u^b}\right) \approx \frac{v^b}{u^b}$; $V_b = V_i = \left(= \sqrt{{u_b}^2 + {v_b}^2 + {w_b}^2}\right)$; these angles represent the angle that the body makes w.r.t 'flight path' or with the direction of the total velocity vector V. In this report we shall assume that these angles are small and may be ignored; in which case the body can be assumed to be aligned with the velocity vector.

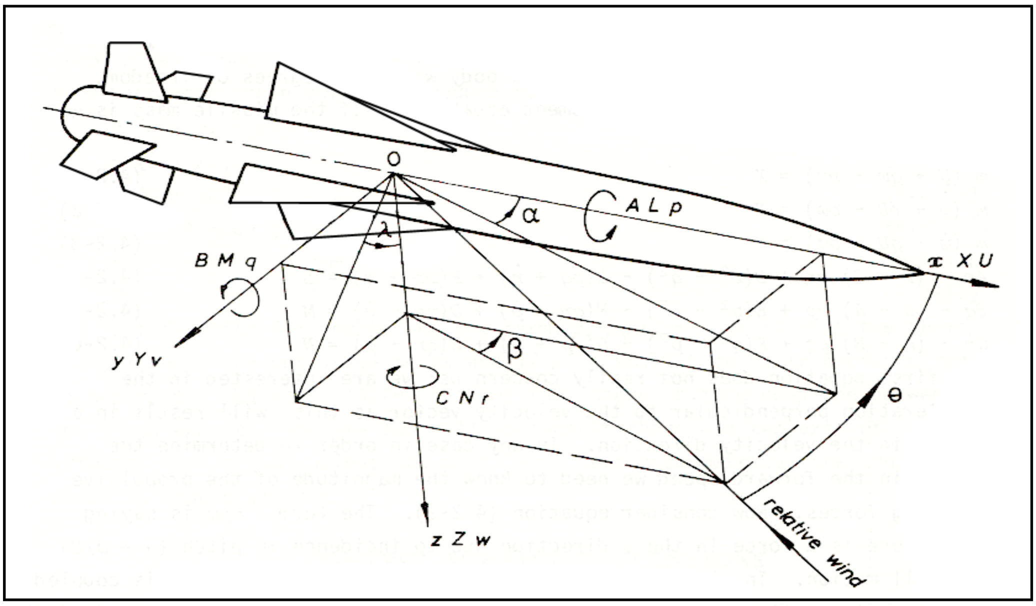

Figure A.1. Aerodynamic variables for a missile

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19. ABSTRACT

In this report mathematical models (2-D Azimuth and Elevation Planes) for multi-party engagement kinematics are derived suitable for developing, implementing and testing modern missile guidance systems. The models developed here are suitable for both conventional and more advanced optimal intelligent guidance schemes including those that arise out of the differential game theory. These models accommodate changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces.

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