Australian Government
Department of Defence
Defence Science and
Technology Organisation

Integrated Navigation, Guidance, and Control of Missile Systems: 3-D Dynamic Model

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DSTO-TR-2805

ABSTRACT

In this report a 3-D mathematical modelfor multi-party engagement kinematics is derived suitable for developing, implementing and testing modern missile guidance systems. The model developed here is suitable for both conventional and more advanced optimal intelligent guidance schemes including those that arise out of the differential game theory. This model accommodates changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces.

RELEASE LIMITATION

Approved for public release

Published by

Weapons Systems Division
DSTO Defence Science and Technology Organisation
PO Box 1500
Edinburgh South Australia 5111 Australia

Telephone: (08) 7389 5555
Fax: (08) 7389 6567

AR-015-532
February 2013

APPROVED FOR PUBLIC RELEASE

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Executive Summary

In the past, linear kinematics models have been used for development and analysis of guidance laws for missile/target engagements. These models were developed in fixed axis system under the assumption that the engagement trajectory does not vary significantly from the collision course geometry. While these models take into account autopilot lags and 'soft' acceleration limits, the guidance commands are applied in fixed axis, and ignore the fact that the missile/target attitude may change significantly during engagement. This latter fact is particularly relevant in cases of engagements where the target implements evasive manoeuvres, resulting in large variations of the engagement trajectory from that of the collision course. A linearised model is convenient for deriving guidance laws (in analytical form), however, the study of their performance characteristics still requires a non-linear model that incorporates changes in body attitudes and implements guidance commands in body axis rather than the fixed axis. In this report, a 3-D mathematical model for multi-party engagement kinematics is derived suitable for developing, implementing and testing modern missile guidance systems. The model developed here is suitable for both conventional and more advanced optimal intelligent guidance, particularly those based on the 'game theory' guidance techniques. These models accommodate changes in vehicle body attitude and other non-linear effects, such as, limits on lateral acceleration and aerodynamic forces. The model presented in this report will be found suitable for computer simulation and analysis of multi-party engagements.

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Farhan A. Faruqi received B.Sc.(Hons) in Mechanical Engineering from the University of Surrey (UK), 1968; M.Sc. in Automatic Control from the University of Manchester Institute of Science and Technology (UK), 1970 and Ph.D from the Imperial College, London University (UK), 1973. He has over 25 years experience in the Aerospace and Defence Industry in UK, Europe and the USA. Prior to joining DSTO in January 1999 he was an Associate Professor at QUT (Australia) 1993-98. Dr. Faruqi is currently the Head of Intelligent Autonomous Systems, Weapons Systems Division, DSTO. His research interests include: Missile Navigation, Guidance and Control, Target Tracking and Precision Pointing Systems, Strategic Defence Systems, Signal Processing, and Optoelectronics.

Contents

Nomenclature

$(\mathbf{i}, \mathbf{j})$:	number of interceptors (pursuers) and targets (evaders) respectively.
$(\mathbf{x_i}, \mathbf{y_i}, \mathbf{z_i})$:	are **x** , **y**, **z**-positions respectively of vehicle **i** in fixed axis.
$(\mathbf{u_i}, \mathbf{v_i}, \mathbf{w_i})$:	are **x, y, z**-velocities respectively of vehicle **i** in fixed axis.
$(\mathbf{a_{x_i}}, \mathbf{a_{y_i}}, \mathbf{a_{z_i}})$:	are **x, y, z**-accelerations respectively of vehicle **i** in fixed axis.
$(\mathbf{x_{ij}}, \mathbf{y_{ij}}, \mathbf{z_{ij}})$:	are **x, y, z**-positions respectively of vehicle **i** w.r.t **j** in fixed axis.
$(\mathbf{u_{ij}}, \mathbf{v_{ij}}, \mathbf{w_{ij}})$:	are **x, y, z**-velocities respectively of vehicle **i** w.r.t **j** in fixed axis.
$(\mathbf{a_{x_{ij}}}, \mathbf{a_{y_{ij}}}, \mathbf{a_{z_{ij}}})$:	are **x, y, z**-accelerations respectively of vehicle **i** w.r.t **j** in fixed axis.
$(\underline{\mathbf{x}}_\mathbf{i}, \underline{\mathbf{u}}_\mathbf{i}, \underline{\mathbf{a}}_\mathbf{i})$:	are **x** ,**y**, **z**-position, velocity and acceleration vectors of vehicle **i** in fixed axis.
$(\underline{\mathbf{x}}_\mathbf{ij}, \underline{\mathbf{u}}_\mathbf{ij}, \underline{\mathbf{a}}_\mathbf{ij})$:	are **x, y, z** - relative position, velocity and acceleration vectors of vehicle **i** w.r.t **j** in fixed axis.
$\mathbf{R_{ij}}$:	separation range of vehicle **i** w.r.t **j** in fixed axis.
$\mathbf{V_{c_{ij}}}$:	closing velocity of vehicle **i** w.r.t **j** in fixed axis.
$\psi_{\mathbf{ij}}, \theta_{\mathbf{ij}}$:	are line-of-sight angle (LOS) of vehicle **i** w.r.t **j** in Azimuth and Elevation planes respectively.
$(\mathbf{a^b_{x_i}}, \mathbf{a^b_{y_i}}, \mathbf{a^b_{z_i}})$:	**x, y, z**-accelerations respectively achieved by vehicle **i** in body axis.
$\left(\mathbf{a^b_{x_{i_d}}}, \mathbf{a^b_{y_{i_d}}}, \mathbf{a^b_{z_{i_d}}}\right)$:	**x, y, z**-accelerations respectively demanded by vehicle **i** in body axis.
$\psi_\mathbf{i}, \theta_\mathbf{i}, \phi_\mathbf{i}$:	are yaw, pitch and roll body (Euler) angles respectively of the **i**th vehicle w.r.t the fixed axis.
$[\mathbf{T^f_b}]_\mathbf{i}$:	is the transformation matrix from body axis to fixed axis.
$\mathbf{V_i}$:	is the velocity of vehicle **i**.
$\tau_{\mathbf{x_i}}$:	autopilot's longitudinal time-constant for vehicle **i**.
$\tau_{\mathbf{y_i}}, \tau_{\mathbf{z_i}}$:	autopilot's lateral time-constant for vehicle **i**.
$\underline{\omega}_{\mathbf{s_{ij}}}$:	line of sight (LOS) rotation vector for vehicles **i** and **j**.
$\underline{\omega}_\mathbf{i} = (\mathbf{P_i}, \mathbf{Q_i}, \mathbf{R_i})$:	body rotation vector for vehicles **i** as seen in fixed axis.
$\underline{\omega}^\mathbf{b}_\mathbf{i} = (\mathbf{p_i}, \mathbf{q_i}, \mathbf{r_i})$:	body rotation vector for vehicles **i** as seen in body axis.
$\underline{\omega}_{\mathbf{i_d}}, \underline{\omega}^\mathbf{b}_{\mathbf{i_d}}$:	demanded body rotation vector for vehicles **i** as seen respectively in fixed and in body axis.

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1. Introduction

In the past [1, 2] linear kinematics models have been used for development and analysis of guidance laws for missile/target engagements. These models were developed in fixed axis under the assumption that the engagement trajectory does not vary significantly from the collision course geometry. While these models take into account autopilot lags and acceleration limits, the guidance commands are applied in fixed axis, and ignore the fact that the missile/target attitude may change significantly during engagement. This latter fact is particularly relevant in cases of engagements where the target implements evasive manoeuvres, resulting in large variations of the engagement trajectory from that of the collision course [3]. Linearised models are convenient for deriving guidance laws (in analytical form), however, the study of their performance characteristics still requires a non-linear model that incorporates changes in body attitudes and implements guidance commands in body axis rather than the fixed axis.

In this report a mathematical model for multi-party engagement kinematics is derived suitable for developing, implementing and testing modern missile guidance systems. The model developed here is suitable for both conventional and more advanced optimal intelligent guidance, particularly those based on the 'game theory' guidance techniques. The model accommodates changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces. Body incidence is assumed to be small and is neglected. The model presented in this report will be found suitable for computer simulation and analysis of multi-party engagements. Sections 2 of this report considers, in some detail, the derivation of engagement dynamics, in section 3, derivations of some of the well known conventional guidance laws, such as, the proportional navigation (PN) and the augmented PN (APN) are given.

2. Development of 3-D Engagement Kinematics Model

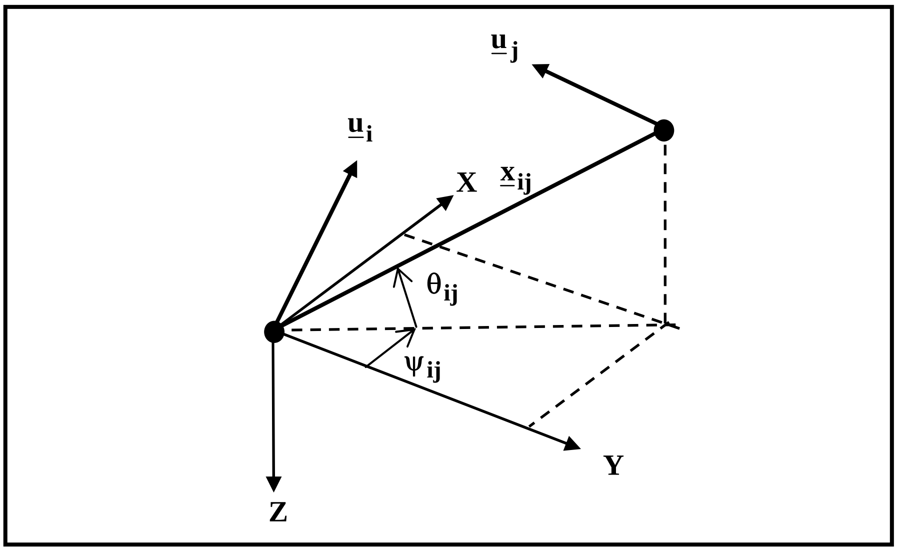

Figure 1-Vehicle Engagement Geometry

2.1 Translational Kinematics for Multi-Vehicle Engagement

A typical 2-vehicle engagement geometry is shown in Figure 1; we shall utilise this to develop the translational kinematics differential equations that relate positions, velocities and accelerations in **x, y, z-** planes of individual vehicles as well as the relative positions, velocities and accelerations. Accordingly, we define the following variables:

(x_i, y_i, z_i): **x ,y, z**-positions respectively of vehicle **i** in fixed axis.

(u_i, v_i, w_i): **x, y, z**-velocities respectively of vehicle **i** in fixed axis.

$(a_{x_i}, a_{y_i}, a_{z_i})$: **x, y, z**-accelerations respectively of vehicle **i** in fixed axis.

The above variables as well as others utilised in this report are functions of time t. The engagement kinematics (i.e. position, velocity and acceleration) involving **n** interceptors (often referred to as pursuers) and **m** targets (referred to as the evaders) $(i = 1,2,...,n+m)$, in fixed axis (e.g. inertial axis) is given by the following set of differential equations:

$$\frac{d}{dt}x_i = u_i \tag{2.1}$$

$$\frac{d}{dt}y_i = v_i \tag{2.2}$$

$$\frac{d}{dt}z_i = w_i \tag{2.3}$$

$$\frac{d}{dt}u_i = a_{x_i} \tag{2.4}$$

$$\frac{d}{dt}v_i = a_{y_i} \tag{2.5}$$

$$\frac{d}{dt}w_i = a_{z_i} \tag{2.6}$$

In order to develop relative kinematics equations for multiple vehicles **i, j** involved in the engagement, $(i : i = 1,2,...,n; j = 1,2,...,m; j \neq i)$, we shall write the relative states:

$x_{ij} = x_i - x_j$: **x**-position of vehicle **i** w.r.t **j** in fixed axis.

$y_{ij} = y_i - j_j$: **y**-position of vehicle **i** w.r.t **j** in fixed axis.

$z_{ij} = z_i - z_j$: **z**-position of vehicle **i** w.r.t **j** in fixed axis.

$u_{ij} = u_i - u_j$: **x**-velocity of vehicle **i** w.r.t **j** in fixed axis.

$v_{ij} = v_i - v_j$: **y**-velocity of vehicle **i** w.r.t **j** in fixed axis.

$w_{ij} = w_i - w_j$: **x**-velocity of vehicle **i** w.r.t **j** in fixed axis.

$a_{x_{ij}} = a_{x_i} - a_{x_j}$: **x**-acceleration of vehicle **i** w.r.t **j** in fixed axis.

$a_{y_{ij}} = a_{y_i} - a_{y_j}$: **y**-acceleration of vehicle **i** w.r.t **j** in fixed axis.

$a_{z_{ij}} = a_{z_i} - a_{z_j}$: **y**-acceleration of vehicle **i** w.r.t **j** in fixed axis.

2.1.1 Vector/Matrix Representation

It will be convenient for model development, to write equations (2.1)-(2.6), in vector notation as follows:

$$\frac{d}{dt}\underline{\mathbf{x}}_{\mathbf{i}} = \underline{\mathbf{u}}_{\mathbf{i}} \qquad (2.7)$$

$$\frac{d}{dt}\underline{\mathbf{u}}_{\mathbf{i}} = \underline{\mathbf{a}}_{\mathbf{i}} \qquad (2.8)$$

Where;

$\underline{\mathbf{x}}_{\mathbf{i}} = [\mathbf{x_i} \quad \mathbf{y_i} \quad \mathbf{z_i}]^{\mathbf{T}}$ **:** is the position vector of vehicle **i** in fixed axis.

$\underline{\mathbf{u}}_{\mathbf{i}} = [\mathbf{u_i} \quad \mathbf{v_i} \quad \mathbf{w_i}]^{\mathbf{T}}$ **:** is the velocity vector of vehicle **i** in fixed axis.

$\underline{\mathbf{a}}_{\mathbf{i}} = [\mathbf{a_{x_i}} \quad \mathbf{a_{y_i}} \quad \mathbf{a_{z_i}}]^{\mathbf{T}}$ **:** is the target acceleration vector of vehicle **i** in fixed axis.

Corresponding differential equations for relative kinematics in vector notation, is given by:

$$\frac{d}{dt}\underline{\mathbf{x}}_{\mathbf{ij}} = \underline{\mathbf{u}}_{\mathbf{ij}} \qquad (2.9)$$

$$\frac{d}{dt}\underline{\mathbf{u}}_{\mathbf{ij}} = \underline{\mathbf{a}}_{\mathbf{i}} - \underline{\mathbf{a}}_{\mathbf{j}} \qquad (2.10)$$

Where:

$\underline{\mathbf{x}}_{\mathbf{ij}} = [\mathbf{x_{ij}} \quad \mathbf{y_{ij}} \quad \mathbf{z_{ij}}]^{\mathbf{T}}$ **:** position vector of vehicle **i** w.r.t **j** in fixed axis.

$\underline{\mathbf{u}}_{\mathbf{ij}} = [\mathbf{u_{ij}} \quad \mathbf{v_{ij}} \quad \mathbf{w_{ij}}]^{\mathbf{T}}$ **:** position vector of vehicle **i** w.r.t **j** in fixed axis.

$\underline{\mathbf{a}}_{\mathbf{ij}} = [\mathbf{a_{x_{ij}}} \quad \mathbf{a_{y_{ij}}} \quad \mathbf{a_{z_{ij}}}]^{\mathbf{T}} = \underline{\mathbf{a}}_{\mathbf{i}} - \underline{\mathbf{a}}_{\mathbf{j}}$ **:** acceleration vector of vehicle **i** w.r.t **j** in fixed axis.

Note: The above formulation admits consideration of engagement where one particular vehicle (interceptor) is fired at another single vehicle (target). In other words we consider one-against-one in a scenario consisting of many vehicles. This consideration can be extended to one-against-many if, for example, $\mathbf{i}$ takes on a single value and $\mathbf{j}$ is allowed to take on a number of different values.

2.2 Constructing Relative Range, Range Rates, Sightline (LOS) Angles and Rates - (Rotational Kinematics)

In this section, we develop rotational kinematics equations involving range and range rates, and sight-line (LOS) angle and rates; measurements of these variables are generally obtained directly from on-board seeker (radar or IR) or derived from on board navigation system or by other indirect means.

2.2.1 Range and Range Rates

The separation range $\mathbf{R_{ij}}$ of vehicles **i** w.r.t **j** may be written as:

$$\left\|\underline{\mathbf{x}}_{\mathbf{ij}}\right\| = \mathbf{R_{ij}} = \left(\mathbf{x_{ij}}^2 + \mathbf{y_{ij}}^2 + \mathbf{z_{ij}}^2\right)^{\frac{1}{2}} = \left(\mathbf{r_{ij}}^2 + \mathbf{z_{ij}}^2\right)^{\frac{1}{2}} = \left(\underline{\mathbf{x}}_{\mathbf{ij}}{}^{\mathbf{T}}\,\underline{\mathbf{x}}_{\mathbf{ij}}\right)^{\frac{1}{2}} \tag{2.11}$$

Expressions for range rate $\dot{\mathbf{R}}_{\mathbf{ij}}$ may be obtained by differentiating the above equations, and is given by:

$$\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{R_{ij}} = \dot{\mathbf{R}}_{\mathbf{ij}} = \frac{\mathbf{x_{ij}u_{ij}} + \mathbf{y_{ij}v_{ij}} + \mathbf{z_{ij}w_{ij}}}{\mathbf{R_{ij}}} = \frac{\left(\underline{\mathbf{x}}_{\mathbf{ij}}{}^{\mathbf{T}}\underline{\mathbf{u}}_{\mathbf{ij}}\right)}{\mathbf{R_{ij}}} \tag{2.12}$$

Another quantity that is often employed in the study of vehicle guidance is the 'closing velocity' $\mathbf{V_{c_{ij}}}$ which is given by:

$$\mathbf{V_{c_{ij}}} = -\dot{\mathbf{R}}_{\mathbf{ij}} \tag{2.13}$$

As noted above, the range and range rate measurements $\mathbf{R_{ij}}, \dot{\mathbf{R}}_{\mathbf{ij}}$ are either directly available or indirectly computed from other available information (or estimated using e.g. a Kalman Filter-KF). To account for errors in these values, we may write:

$$\hat{\mathbf{R}}_{\mathbf{ij}} = \mathbf{R_{ij}} + \Delta\mathbf{R_{ij}} \tag{2.14}$$

$$\hat{\dot{\mathbf{R}}}_{\mathbf{ij}} = \dot{\mathbf{R}}_{\mathbf{ij}} + \Delta\dot{\mathbf{R}}_{\mathbf{ij}} \tag{2.15}$$

Where:

$\hat{\mathbf{R}}_{\mathbf{ij}}$ **:** is the estimated/measured value of the relative range.

$\hat{\dot{\mathbf{R}}}_{\mathbf{ij}}$ **:** is the estimated/measured value of the relative range rate.

$\Delta\mathbf{R_{ij}}$ **:** is the measurement error in relative range.

$\Delta\dot{\mathbf{R}}_{\mathbf{ij}}$ **:** is the measurement error in relative range rate.

2.2.2 Sightline Rates

The sight line rotation vector $\underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}}$ (see Figure 2) is related to the relative range and velocity $\underline{\mathbf{x}}_{\mathbf{ij}}, \underline{\mathbf{u}}_{\mathbf{ij}}$ as follows:

$$\underline{\mathbf{u}}_{\mathbf{ij}} = \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{x}}_{\mathbf{ij}} \tag{2.16}$$

Where:

$\underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} = \left\lfloor \omega_{1_{ij}} \quad \omega_{2_{ij}} \quad \omega_{3_{ij}} \right\rfloor^{\mathbf{T}}$ **:** is the LOS rotation vector of vehicle **i** w.r.t **j** as defined in (as seen in) fixed axis.

It is well known that the vector triple product which is the cross product of a vector with the result of another cross product, is related to the dot product by the following formula [4]: $\underline{\mathbf{a}} \times \underline{\mathbf{b}} \times \underline{\mathbf{c}} = \underline{\mathbf{b}}\left(\underline{\mathbf{a}}.\underline{\mathbf{c}}\right) - \underline{\mathbf{c}}\left(\underline{\mathbf{a}}.\underline{\mathbf{b}}\right)$. Taking the cross-product of both sides of (2.16) by $\underline{\mathbf{x}}_{\mathbf{ij}}$ and applying this rule, we get

$$\underline{\mathbf{x}}_{ij} \times \underline{\mathbf{u}}_{ij} = \underline{\mathbf{x}}_{ij} \times \left(\underline{\boldsymbol{\omega}}_{s_{ij}} \times \underline{\mathbf{x}}_{ij} \right) = \underline{\boldsymbol{\omega}}_{s_{ij}} \left(\underline{\mathbf{x}}_{ij} \cdot \underline{\mathbf{x}}_{ij} \right) - \underline{\mathbf{x}}_{ij} \left(\underline{\mathbf{x}}_{ij} \cdot \underline{\boldsymbol{\omega}}_{s_{ij}} \right) \tag{2.17}$$

Since $\underline{\mathbf{x}}_{ij}$ and $\underline{\boldsymbol{\omega}}_{s_{ij}}$ are mutually orthogonal, therefore $\left(\underline{\mathbf{x}}_{ij} \cdot \underline{\boldsymbol{\omega}}_{s_{ij}} \right) = \mathbf{0}$; hence:

$$\underline{\mathbf{x}}_{ij} \times \underline{\mathbf{u}}_{ij} = \underline{\boldsymbol{\omega}}_{s_{ij}} \left(\underline{\mathbf{x}}_{ij} \cdot \underline{\mathbf{x}}_{ij} \right) = \underline{\boldsymbol{\omega}}_{s_{ij}} \left(\underline{\mathbf{x}}_{ij}^{\mathbf{T}} \underline{\mathbf{x}}_{ij} \right)$$

➔

$$\underline{\boldsymbol{\omega}}_{s_{ij}} = \frac{\underline{\mathbf{x}}_{ij} \times \underline{\mathbf{u}}_{ij}}{\underline{\mathbf{x}}_{ij}^{\mathbf{T}} \underline{\mathbf{x}}_{ij}} = \frac{1}{\underline{\mathbf{x}}_{ij}^{\mathbf{T}} \underline{\mathbf{x}}_{ij}} \begin{bmatrix} 0 & w_{ij} & -v_{ij} \\ -w_{ij} & 0 & u_{ij} \\ v_{ij} & -u_{ij} & 0 \end{bmatrix} \begin{bmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{bmatrix} \cdots$$
$$= \frac{1}{\underline{\mathbf{x}}_{ij}^{\mathbf{T}} \underline{\mathbf{x}}_{ij}} \begin{bmatrix} \left(y_{ij} w_{ij} - z_{ij} v_{ij} \right) \\ \left(z_{ij} u_{ij} - x_{ij} w_{ij} \right) \\ \left(x_{ij} v_{ij} - y_{ij} u_{ij} \right) \end{bmatrix} \tag{2.18}$$

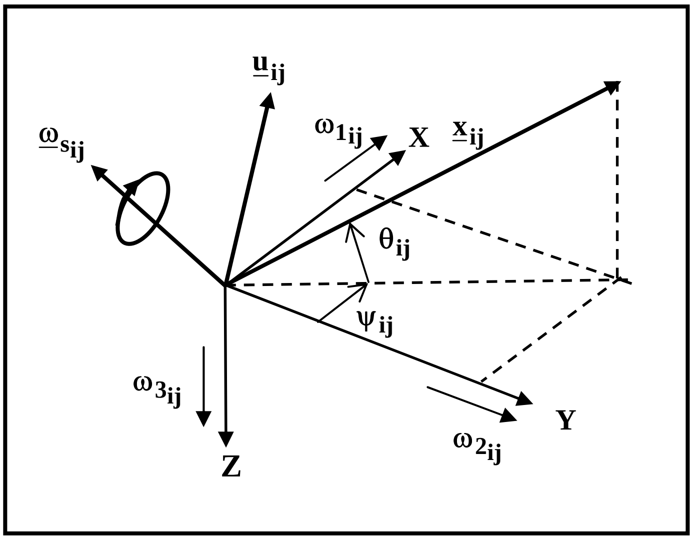

Figure 2. Line of Sight Rotation

If sightline rate values are required in body frame then equation (2.18) has to be transformed to body axis to obtain sightline rates in body axis. The measurement $\hat{\boldsymbol{\omega}}_{s_{ij}}$ obtained from the seeker used to construct the guidance commands is given by:

$$\underline{\hat{\boldsymbol{\omega}}}_{s_{ij}} = \underline{\boldsymbol{\omega}}_{s_{ij}} + \Delta \underline{\boldsymbol{\omega}}_{s_{ij}} \tag{2.19}$$

Where:

$\Delta \underline{\boldsymbol{\omega}}_{s_{ij}}$ **:** seeker LOS rate measurement error.

The above relationships (2.11)-(2.19) will also be referred to as the ***seeker model***.

2.3 Vehicle Navigation Model

The vehicle navigation part of the model is concerned with developing equations that allow the angular rotation, of the vehicle body, to be generated and subsequently computing the elements of the transformation (direction cosine) matrix. We shall utilise the quaternion algebra [5] to achieve this.

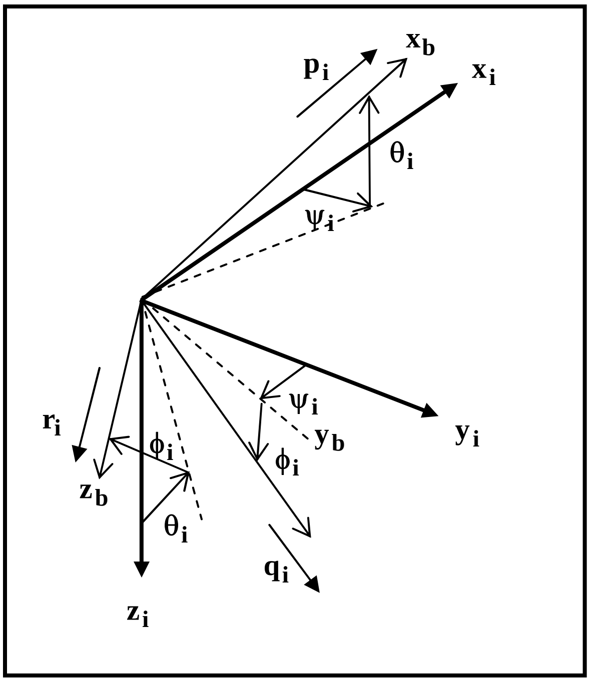

Figure 3. Axis System Rotation Convention

Let us define the following:

$\left(a^b_{x_i}, a^b_{y_i}, a^b_{z_i}\right)$: **x, y, z-**acceleration achieved by vehicle **i** in its body axis.

The transformation matrix from <u>fixed to body</u> axis $\left[T^b_f\right]_i$, for vehicle **i** is given by (see Figure 3):

$$\begin{bmatrix} a^b_{x_i} \\ a^b_{y_i} \\ a^b_{z_i} \end{bmatrix} = \begin{bmatrix} (c\theta_i \; c\psi_i) & (c\theta_i \; s\psi_i) & (-s\theta_i) \\ (s\phi_i \; s\theta_i \; c\psi_i - c\phi_i \; s\psi_i) & (s\phi_i \; s\theta_i \; s\psi_i + c\phi_i \; c\psi_i) & (s\phi_i \; c\theta_i) \\ (c\phi_i \; s\theta_i \; c\psi_i + s\phi_i \; s\psi_i) & (c\phi_i \; s\theta_i \; s\psi_i - s\phi_i \; c\psi_i) & (c\phi_i \; c\theta_i) \end{bmatrix} \begin{bmatrix} a_{x_i} \\ a_{y_i} \\ a_{z_i} \end{bmatrix}$$

Abbreviations **s** and **c** are used for sin and cos of angles respectively.

This equation may also be written as:

$$\begin{bmatrix} a^b_{x_i} \\ a^b_{y_i} \\ a^b_{z_i} \end{bmatrix} = \left[T^b_f\right]_i \begin{bmatrix} a_{x_i} \\ a_{y_i} \\ a_{z_i} \end{bmatrix} \quad (2.20)$$

In vector/matrix notation this equation along with its companion (inverse) transformation, may be written as:

$$\underline{a}^b_i = \left[T^b_f\right]_i \underline{a}_i \quad (2.21)$$

$$\underline{a}_i = \left[T^f_b\right]_i \underline{a}^b_i \quad (2.22)$$

Where:

$(\psi_i, \theta_i, \phi_i)$: are (Euler) angles of the **i**th vehicle w.r.t the fixed axis.

$\underline{a}^b_i = \left[a^b_{x_i} \quad a^b_{y_i} \quad a^b_{z_i}\right]^T$ **:** is acceleration vector of vehicle **i** in its body axis.

$\left[T^f_b\right]_i = \left[T^b_f\right]^T_i = \left[T^b_f\right]^{-1}_i$ **:** is the transformation matrix from body to fixed axis.

2.3.1 Application of Quaternion to Navigation

A fuller exposition on quaternion algebra is given in [5]; in this section the main results are utilised for the navigation model for constructing the transformation matrix. We define the following quantities, referred to as the ***quaternions***, for vehicle **i** as follows:

$$q_{1_i} = \cos\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\cos\frac{\psi_i}{2} + \sin\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\sin\frac{\psi_i}{2} \quad (2.23)$$

$$q_{2_i} = \sin\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\cos\frac{\psi_i}{2} - \cos\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\sin\frac{\psi_i}{2} \quad (2.24)$$

$$q_{3_i} = \cos\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\cos\frac{\psi_i}{2} + \sin\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\sin\frac{\psi_i}{2} \quad (2.25)$$

$$q_{4_i} = \cos\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\sin\frac{\psi_i}{2} - \sin\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\cos\frac{\psi_i}{2} \quad (2.26)$$

It can be shown that the transformation matrix $\left[T^b_f\right]_i$ for the **i**th vehicle may be written as:

$$\left[T^b_f\right]_i = \begin{bmatrix} t_{11_i} & t_{12_i} & t_{13_i} \\ t_{21_i} & t_{22_i} & t_{23_i} \\ t_{31_i} & t32 & t_{33_i} \end{bmatrix} \quad (2.27)$$

Where: the elements of $\left[T^b_f\right]_i = \left[T^b_f\left(\underline{q}_i\right)\right]$ are functions of the quaternions are given by the following relations:

$$t_{11_i} = \left(q_{1_i}{}^2 + q_{2_i}{}^2 - q_{3_i}{}^2 - q_{4_i}{}^2\right) \quad (2.28)$$

$$t_{12_i} = 2\left(q_{2_i}q_{3_i} + q_{1_i}q_{4_i}\right) \quad (2.29)$$

$$t_{13_i} = 2\left(q_{2_i} q_{4_i} - q_{1_i} q_{3_i}\right) \quad (2.30)$$
$$t_{21_i} = 2\left(q_{2_i} q_{3_i} - q_{1_i} q_{4_i}\right) \quad (2.31)$$
$$t_{22_i} = \left({q_{1_i}}^2 - {q_{2_i}}^2 + {q_{3_i}}^2 - {q_{4_i}}^2\right) \quad (2.32)$$
$$t_{23_i} = 2\left(q_{3_i} q_{4_i} + q_{1_i} q_{2_i}\right) \quad (2.33)$$
$$t_{31_i} = 2\left(q_{2_i} q_{4_i} + q_{1_i} q_{3_i}\right) \quad (2.34)$$
$$t_{32_i} = 2\left(q_{3_i} q_{4_i} - q_{1_i} q_{2_i}\right) \quad (2.35)$$
$$t_{33_i} = \left({q_{1_i}}^2 - {q_{2_i}}^2 - {q_{3_i}}^2 + {q_{4_i}}^2\right) \quad (2.36)$$

The time-evolution of quaternion is given by the following differential equation:

$$\frac{d}{dt}\begin{bmatrix} q_{1_i} \\ q_{2_i} \\ q_{3_i} \\ q_{4_i} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -p_i & -q_i & -r_i \\ p_i & 0 & r_i & -q_i \\ q_i & -r_i & 0 & p_i \\ r_i & q_i & -p_i & 0 \end{bmatrix}\begin{bmatrix} q_{1_i} \\ q_{2_i} \\ q_{3_i} \\ q_{4_i} \end{bmatrix} \quad (2.37)$$

In vector notation equation (2.37) may be written as:

$$\frac{d}{dt}\underline{q}_i = \left[\Omega_i^b\right]\underline{q}_i \quad (2.38)$$

Where:

$$\left[\Omega_i^b\right] = \frac{1}{2}\begin{bmatrix} 0 & -p_i & -q_i & -r_i \\ p_i & 0 & r_i & -q_i \\ q_i & -r_i & 0 & p_i \\ r_i & q_i & -p_i & 0 \end{bmatrix}$$

$\underline{q}_i = \left[q_{1_i} \quad q_{2_i} \quad q_{3_i} \quad q_{4_i}\right]^T$ **:** is the quaternion vector for vehicle **i**.

$\underline{\omega}_i^b = \left[p_i \quad q_i \quad r_i\right]^T$ **:** is the rotation vector of vehicle **i** w.r.t to the fixed axis as seen in the body axis (also referred to as body rate vector).

The Euler angles, in terms of the elements of the transformation matrix, may be written as:

$$\phi_i = \tan^{-1}\left(\frac{t_{23_i}}{t_{33_i}}\right) \quad (2.39)$$
$$\theta_i = \sin^{-1}\left(-t_{13_i}\right) \quad (2.40)$$
$$\psi_i = \tan^{-1}\left(\frac{t_{12_i}}{t_{11_i}}\right) \quad (2.41)$$

$\theta_i \neq 90^o$

2.3.2 Body Incidence Angles and Flight Path Angles

he vehicle (absolute) velocity in fixed axis (which is the same as the absolute velocity in body axis) is given by:

$$\mathbf{V_i} = \left(\mathbf{u_i}^2 + \mathbf{v_i}^2 + \mathbf{w_i}^2\right)^{\frac{1}{2}} = \left(\underline{\mathbf{u}}_\mathbf{i}^\mathbf{T}\underline{\mathbf{u}}_\mathbf{i}\right)^{\frac{1}{2}} = \left(\underline{\mathbf{u}}_\mathbf{i}^{\mathbf{b}\,\mathbf{T}}\underline{\mathbf{u}}_\mathbf{i}^\mathbf{b}\right)^{\frac{1}{2}} = \mathbf{V_i^b} \tag{2.42}$$

Where:

$\underline{\mathbf{u}}_\mathbf{i}^\mathbf{b} = \left[\mathbf{u_i^b} \quad \mathbf{v_i^b} \quad \mathbf{w_i^b}\right]^\mathbf{T}$ **:** is the velocity vector of vehicle **i** in body axis.

$\mathbf{V_i^b} = \mathbf{V_i}$ **:** is the vehicle velocity in body axis.

Given that the body incidence angles in pitch and yaw are $(\alpha_\mathbf{i}, \beta_\mathbf{i})$, the ***flight path*** angles in pitch and yaw (=angle that the velocity vector makes with the fixed axis) are respectively $(\theta_\mathbf{i} - \alpha_\mathbf{i})$ and $(\psi_\mathbf{i} - \beta_\mathbf{i})$.

Where:

$\alpha_\mathbf{i} = \mathbf{tan}^{-1}\dfrac{\mathbf{w_i^b}}{\mathbf{u_i^b}}$ **:** is the body pitch incidence angle.

$\beta_\mathbf{i} = \mathbf{tan}^{-1}\dfrac{\mathbf{v_i^b}}{\mathbf{u_i^b}}$ **:** is the body azimuth incidence (side-slip angle).

Assuming that $(\mathbf{v_i^b}, \mathbf{w_i^b}) << \mathbf{u_i^b}$ thus lends justification to the assumption that $(\alpha_\mathbf{i}, \beta_\mathbf{i})$ are small. Furthermore differentiating the expressions for $(\alpha_\mathbf{i}, \beta_\mathbf{i})$ and simplifying gives us:

$$\dot{\alpha}_\mathbf{i} = \frac{\dot{\mathbf{w}}_\mathbf{i}^\mathbf{b}\mathbf{u_i^b} - \dot{\mathbf{u}}_\mathbf{i}^\mathbf{b}\mathbf{w_i^b}}{\left(\mathbf{u_i^b}^2 + \mathbf{w_i^b}^2\right)} \tag{2.43}$$

$$\dot{\beta}_\mathbf{i} = \frac{\dot{\mathbf{v}}_\mathbf{i}^\mathbf{b}\mathbf{u_i^b} - \dot{\mathbf{u}}_\mathbf{i}^\mathbf{b}\mathbf{v_i^b}}{\left(\mathbf{u_i^b}^2 + \mathbf{v_i^b}^2\right)} \tag{2.44}$$

For $(\dot{\mathbf{v}}_\mathbf{i}^\mathbf{b}, \dot{\mathbf{w}}_\mathbf{i}^\mathbf{b}, \dot{\mathbf{u}}_\mathbf{i}^\mathbf{b}) << \mathbf{u_i^b}$, we get $(\dot{\alpha}_\mathbf{i}, \dot{\beta}_\mathbf{i}) \approx \mathbf{0}.$ In this report we shall assume that the incidence angles $(\alpha_\mathbf{i}, \beta_\mathbf{i})$ and the rates $(\dot{\alpha}_\mathbf{i}, \dot{\beta}_\mathbf{i})$ are small and hence can be ignored; and the vehicle body may be assumed to be aligned to the velocity vector.

2.3.3 Computing Body Rates $(\mathbf{p_i}, \mathbf{q_i}, \mathbf{r_i})$

We now consider equations (A1.1)-(A1.3), from Appendix-1 for the **i**[th] vehicle, which we write as:

$$\mathbf{a_{x_i}^b} = \dot{\mathbf{u}}_\mathbf{i}^\mathbf{b} + \mathbf{qw_i^b} - \mathbf{rv_i^b} \tag{2.45}$$

$$\mathbf{a_{y_i}^b} = \dot{\mathbf{v}}_\mathbf{i}^\mathbf{b} + \mathbf{ru_i^b} - \mathbf{pw_i^b} \tag{2.46}$$

$$\mathbf{a_{z_i}^b} = \dot{\mathbf{w}}_\mathbf{i}^\mathbf{b} + \mathbf{pv_i^b} - \mathbf{qu_i^b} \tag{2.47}$$

In matrix notation equations (2.45)-(2.47) may be written as:

$$\begin{bmatrix} \mathbf{a^b_{x_i}} \\ \mathbf{a^b_{y_i}} \\ \mathbf{a^b_{y_i}} \end{bmatrix} = \begin{bmatrix} \mathbf{\dot{u}^b_i} \\ \mathbf{\dot{v}^b_i} \\ \mathbf{\dot{w}^b_i} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{r_i} & \mathbf{q_i} \\ \mathbf{r_i} & \mathbf{0} & -\mathbf{p_i} \\ -\mathbf{q_i} & \mathbf{p_i} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u^b_i} \\ \mathbf{v^b_i} \\ \mathbf{w^b_i} \end{bmatrix} \tag{2.48}$$

This equation is of the form:

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} = \underline{\dot{\mathbf{u}}}^{\mathbf{b}}_{\mathbf{i}} + \underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \tag{2.49}$$

Note: As pointed out earlier it is assumed that the missile body is aligned with the velocity vector (section 2.3.2); moreover if $\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}}$ is regarded as the lateral acceleration acting on the velocity vector then $\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}}, \underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}}, \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}$ can be assumed to be mutually orthogonal. i.e.: $\left(\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} . \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}\right) = \left(\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} . \underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}}\right) = \left(\underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} . \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}\right) = \mathbf{0}$.

Taking the cross product of equation (2.49) with $\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}$; applying the triple cross-product rule, and noting the fact that $\left(\underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} . \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}\right) = \mathbf{0}$, we get:

$$\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} = \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\dot{\mathbf{u}}}^{\mathbf{b}}_{\mathbf{i}} + \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} = \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\dot{\mathbf{u}}}^{\mathbf{b}}_{\mathbf{i}} + \underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} \left(\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \cdot \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}\right) \tag{2.50}$$

➔

$$\underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} = \frac{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}}}{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \cdot \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} - \frac{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\dot{\mathbf{u}}}^{\mathbf{b}}_{\mathbf{i}}}{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \cdot \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} \tag{2.51}$$

For $\left(\mathbf{\dot{v}^b_i}, \mathbf{\dot{w}^b_i}, \mathbf{\dot{u}^b_i}\right)$ **and** $\left(\mathbf{v^b_i}, \mathbf{w^b_i}\right) << \mathbf{u^b_i}$, it follows that the second term on the RHS of (2.51):

$$\frac{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\dot{\mathbf{u}}}^{\mathbf{b}}_{\mathbf{i}}}{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \cdot \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} = \frac{\left[\left(\mathbf{v^b_i \dot{w}^b_i - w^b_i \dot{v}^b_i}\right) \quad \left(\mathbf{w^b_i \dot{u}^b_i - u^b_i \dot{w}^b_i}\right) \quad \left(\mathbf{u^b_i \dot{v}^b_i - v^b_i \dot{u}^b_i}\right)\right]^{\mathbf{T}}}{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \cdot \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} \approx \underline{\mathbf{0}} \tag{2.52}$$

➔

$$\underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} = \frac{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}}}{\underline{\mathbf{u}}^{\mathbf{b}^{\mathbf{T}}}_{\mathbf{i}} \ \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} = \frac{\left[\left(\mathbf{v^b_i a^b_{z_i} - w^b_i a^b_{y_i}}\right) \quad \left(\mathbf{w^b_i a^b_{x_i} - u^b_i a^b_{z_i}}\right) \quad \left(\mathbf{u^b_i a^b_{y_i} - v^b_i a^b_{x_i}}\right)\right]^{\mathbf{T}}}{\underline{\mathbf{u}}^{\mathbf{b}^{\mathbf{T}}}_{\mathbf{i}} \ \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} \tag{2.53}$$

2.4 Vehicle Autopilot Dynamics

Assuming a first order lag for the autopilot, we may write for vehicle $\mathbf{i}$:

$$\frac{\mathbf{d}}{\mathbf{dt}} \mathbf{a^b_{x_i}} = -\tau_{\mathbf{x_i}} \ \mathbf{a^b_{x_i}} + \tau_{\mathbf{x_i}} \ \mathbf{a^b_{x_{i_d}}} \tag{2.54}$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \mathbf{a^b_{y_i}} = -\tau_{\mathbf{y_i}} \ \mathbf{a^b_{y_i}} + \tau_{\mathbf{y_i}} \ \mathbf{a^b_{y_{i_d}}} \tag{2.55}$$

$$\frac{\mathbf{d}}{\mathbf{dt}} \mathbf{a^b_{z_i}} = -\tau_{\mathbf{z_i}} \ \mathbf{a^b_{z_i}} + \tau_{\mathbf{z_i}} \ \mathbf{a^b_{z_{i_d}}} \tag{2.56}$$

In vector/matrix notation equations (2.54)-(2.56) may be written as:

$$\frac{d}{dt}\underline{a}_i^b = [-\Lambda_i]\underline{a}_i^b + [\Lambda_i]\underline{a}_{i_d}^b \qquad (2.57)$$

Where:

τ_{x_i} : Vehicle **i** autopilot's longitudinal time-constant.

τ_{y_i} : Vehicle **i** autopilot's (lateral) yaw-plane time-constant.

τ_{z_i} : Vehicle **i** autopilot's (lateral) pitch-plane time-constant.

$$[\Lambda_i] = \begin{bmatrix} \tau_{x_i} & 0 & 0 \\ 0 & \tau_{y_i} & 0 \\ 0 & 0 & \tau_{z_i} \end{bmatrix}$$

$a_{x_{i_d}}^b$: **x**-acceleration demanded by vehicle **i** in its body axis.

$a_{y_{i_d}}^b$: **y**-acceleration demanded by vehicle **i** in its body axis.

$a_{z_{i_d}}^b$: **z**-acceleration demanded by vehicle **i** in its body axis.

$\underline{a}_{i_d}^b = \begin{bmatrix} a_{x_{i_d}}^b & a_{y_{i_d}}^b & a_{z_{i_d}}^b \end{bmatrix}^T$: is the demanded missile acceleration (command input) vector in body axis.

2.5 Aerodynamic Considerations

Detailed consideration of the effects of the aerodynamic forces is contained in Appendix-1. Generally, the longitudinal acceleration $a_{x_{i_d}}^b = \frac{\delta T_i - \delta D_i}{m_i}$ of a missile is not varied in response to the guidance commands and may be assumed to be zero. However, the nominal values, which define the steady state flight: $\bar{a}_{x_i}^b = \frac{(\bar{T} - \bar{D})}{m} - g\sin\bar{\theta}$, $\bar{a}_{z_i}^b = \frac{\bar{Y}}{m} + g\cos\bar{\theta}\sin\bar{\phi}$ and $\bar{a}_{z_i}^b = \frac{\bar{Z}}{m} + g\cos\bar{\theta}\cos\bar{\phi}$ may change due to changes in flight conditions and need to be included in the simulation model; this is shown in the block diagram Figure A.1. The variations in lateral accelerations $\delta a_y^b = \frac{\delta Y}{m} = \delta\tilde{Y}$, $\delta a_z^b = \frac{\delta Z}{m} = \delta\tilde{Z}$, on the other hand provide the necessary control effort required for guidance; the limits on these may be implemented as shown in Appendix-1, that is: $\left\| a^b{}_{y_{i_d}} \right\| \le \mu\, a_{y_{max}}$ and $\left\| a^b{}_{z_{i_d}} \right\| \le \mu\, a_{z_{max}}$.

3. Conventional Guidance Laws

3.1 Proportional Navigation (PN) Guidance

There are at least three versions of PN guidance laws that the author is aware of; in this section we consider two of these, which are (for interceptor **i** -the pursuer against a target **j** – the evader):

3.1.1 Version 1 (PN-1):

This implementation is based on the principle that the demanded body rate of the attacker **i** is proportional to LOS rate to the target **j**(see Figure 1); that is:

$$\underline{\boldsymbol{\omega}}_{\mathbf{i_d}} = [\mathbf{N}]_{\mathbf{i}}\, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \tag{3.1}$$

Where:

$\underline{\boldsymbol{\omega}}_{\mathbf{i_d}} = \left[\mathbf{P_{i_d}} \quad \mathbf{Q_{i_d}} \quad \mathbf{R_{i_d}}\right]^{\mathbf{T}}$ **:** is the demanded body rotation vector of vehicle **i** in the fixed axis.

$[\mathbf{N}]_{\mathbf{i}} = \mathbf{diag}\left(\mathbf{N_{1_i}} \quad \mathbf{N_{2_i}} \quad \mathbf{N_{3_i}}\right)$**:** are navigation constants attached to respective demand channels. If the longitudinal acceleration is not possible (as is the case in most missiles) then $\mathbf{N_{1_i}} = \mathbf{0}$.

The acceleration demanded of vehicle **i** is given by:

$$\underline{\mathbf{a}}_{\mathbf{i_d}} = \underline{\boldsymbol{\omega}}_{\mathbf{i_d}} \times \underline{\mathbf{u}}_{\mathbf{i}} = [\mathbf{N}]_{\mathbf{i}}\, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{u}}_{\mathbf{i}} \tag{3.2}$$

Since the guidance commands are applied in body axis, we need to transform equation (3.2) to body axis, thus:

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} = \left[\mathbf{T_f^b}\right]_{\mathbf{i}}\left([\mathbf{N}]_{\mathbf{i}}\, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{u}}_{\mathbf{i}}\right) \tag{3.3}$$

Assuming that the longitudinal acceleration in response to the guidance commands is zero, we get:

$$\mathbf{a}^{\mathbf{b}}_{\mathbf{x_{i_d}}} = \frac{\boldsymbol{\delta}\mathbf{T_i} - \boldsymbol{\delta}\mathbf{D_i}}{\mathbf{m_i}} = \mathbf{0} \tag{3.4}$$

3.1.2 Version 2 (PN-2):

This implementation is based on the principle that the demanded lateral acceleration of the attacker **i** is proportional to the acceleration normal to the LOS acceleration to target **j**, caused by the LOS rotation. Now, the LOS acceleration is given by:

$$\underline{\mathbf{a}}_{\mathbf{n_{ij}}} = \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{u}}_{\mathbf{ij}} \tag{3.5}$$

➔

$$\underline{\mathbf{a}}_{\mathbf{i_d}} = [\mathbf{N}]_{\mathbf{i}}\, \underline{\mathbf{a}}_{\mathbf{n_{ij}}} = [\mathbf{N}]_{\mathbf{i}}\, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{u}}_{\mathbf{ij}}$$

Transforming to body axis, gives us:

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} = \left[\mathbf{T}^{\mathbf{b}}_{\mathbf{f}}\right]_{\mathbf{i}}\left(\left[\mathbf{N}\right]_{\mathbf{i}}\ \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} \times \underline{\mathbf{u}}_{\mathbf{ij}}\right) \tag{3.6}$$

Once again, assuming that the longitudinal acceleration in response to the guidance commands is zero, we get:

$$\mathbf{a}^{\mathbf{b}}_{\mathbf{x_{i_d}}} = \frac{\delta\mathbf{T_i} - \delta\mathbf{D_i}}{\mathbf{m_i}} = \mathbf{0} \tag{3.7}$$

Where:

$\underline{\mathbf{a}}_{\mathbf{n_{ji}}}$ **:** is the normal LOS acceleration.

3.2 Augmented Proportional Navigation (APN) Guidance

Finally, a variation of the PN guidance law is the APN that includes the influence of the target acceleration, can be implemented as follows:

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} = \left[\mathbf{PNG}\left(\mathbf{N}, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}}\right)\right] + \left[\mathbf{T}^{\mathbf{b}}_{\mathbf{f}}\right]_{\mathbf{i}}\left(\left[\overline{\mathbf{N}}\right]\underline{\mathbf{a}}_{\mathbf{j}}\right) \tag{3.8}$$

Where:

$\left[\overline{\mathbf{N}}\right]$**:** is the (target) acceleration navigation constant.

$\left(\mathbf{PNG}\right)$**:** is the proportional navigation guidance law given in (3.1)-(3.7)

4. Overall State Space Model

The overall non-linear state space model (e.g. for APN guidance) that can be used for sensitivity studies and for non-linear or Monte-Carlo analysis is given below:

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{x}}_{\mathbf{ij}} = \underline{\mathbf{u}}_{\mathbf{ij}} \tag{4.1}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{u}}_{\mathbf{ij}} = \left[\mathbf{T}^{\mathbf{f}}_{\mathbf{b}}\right]_{\mathbf{i}}\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} - \underline{\mathbf{a}}_{\mathbf{j}} \tag{4.2}$$

$$\underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}} = \frac{\underline{\mathbf{x}}_{\mathbf{ij}} \times \underline{\mathbf{u}}_{\mathbf{ij}}}{\underline{\mathbf{x}}_{\mathbf{ij}}^{\mathbf{T}}\underline{\mathbf{x}}_{\mathbf{ij}}} \tag{4.3}$$

$$\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} = \left[\mathbf{PNG}\left(\mathbf{N}, \underline{\boldsymbol{\omega}}_{\mathbf{s_{ij}}}\right)\right] + \left[\mathbf{T}^{\mathbf{b}}_{\mathbf{f}}\right]_{\mathbf{i}}\left[\overline{\mathbf{N}}\right]\underline{\mathbf{a}}_{\mathbf{j}} \tag{4.4}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} = \left[-\Lambda_{\mathbf{i}}\right]\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}} + \left[\Lambda_{\mathbf{i}}\right]\underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i_d}} \tag{4.5}$$

$$\underline{\boldsymbol{\omega}}^{\mathbf{b}}_{\mathbf{i}} = \frac{\underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}} \times \underline{\mathbf{a}}^{\mathbf{b}}_{\mathbf{i}}}{\underline{\mathbf{u}}^{\mathbf{b}\,\mathbf{T}}_{\mathbf{i}}\ \underline{\mathbf{u}}^{\mathbf{b}}_{\mathbf{i}}} \tag{4.6}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\underline{\mathbf{q}}_{\mathbf{i}} = \left[\Omega^{\mathbf{b}}_{\mathbf{bf}}\right]\underline{\mathbf{q}}_{\mathbf{i}} \tag{4.7}$$

The overall state space model that can be implemented on the computer is given in Table A.1, and the block-diagram is shown in Figure A.1.

5. Conclusions

In this report mathematical model is derived for multi-vehicle guidance, navigation and control suitable for developing, implementing and testing modern missile guidance systems. The model allows for incorporating changes in body attitude in addition to autopilot lags, vehicle acceleration limits and aerodynamic effects. This model will be found suitable for studying the performance of both the conventional and the modern guidance such as those that arise of game theory and intelligent control theory. The following are considered to be the main contribution of this report:

- A 3-DOF multi-vehicle engagement model is derived for the purposes of developing, testing, and carrying out guidance performance studies,
- The model incorporates non-linear effects including large changes in vehicle body attitude, autopilot lags, acceleration limits and aerodynamic effects,
- The model can easily be adapted for multi-run non-linear analysis of guidance performance and for undertaking Monte-Carlo analysis.

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Acknowledgements

The author would like to acknowledge the contribution of Mr. Jim Repo and Mr. Arvind Rajagopalan for their helpful suggestion and in assisting in the development of computer simulation programming.

Appendix A

Table A.1: State Space Dynamics Model for Navigation, Seeker, Guidance and Autopilot.

	ALGORITHM	MODULE NAME
1	$\frac{d}{dt}\underline{x}_i = \underline{u}_i$ $\frac{d}{dt}\underline{u}_i = \underline{a}_i$ $\underline{x}_{ij} = \underline{x}_i - \underline{x}_j$ $\underline{u}_{ij} = \underline{u}_i - \underline{u}_j$ $\underline{a}_{ij} = \underline{a}_i - \underline{a}_j$	**Translational Kinematics**
2	$R_{ij} = \left(\underline{x}_{ij}{}^T \underline{x}_{ij}\right)^{\frac{1}{2}}$ $\hat{R}_{ij} = R_{ij} + \Delta R_{ij}$ $\frac{d}{dt}R_{ij} = \dot{R}_{ij} = \frac{\left(\underline{x}_{ij}{}^T \underline{u}_{ij}\right)}{R_{ij}}$ $\dot{\hat{R}}_{ij} = \dot{R}_{ij} + \Delta\dot{R}_{ij}$ $V_{c_{ij}} = -\dot{\hat{R}}_{ij}$ $\underline{\omega}_{s_{ij}} = \frac{\underline{x}_{ij} \times \underline{u}_{ij}}{\underline{x}_{ij}{}^T \underline{x}_{ij}}$ $\hat{\underline{\omega}}_{s_{ij}} = \underline{\omega}_{s_{ij}} + \Delta\underline{\omega}_{s_{ij}}$	**Rotational Kinematics (Seeker Model)**
3	3.1. $\underline{a}^b_{i_d} = \left[T^b_f\right]_i [N]_i \, \hat{\underline{\omega}}_{s_{ij}} \times \underline{u}_i$ 3.2. $\underline{a}^b_{i_d} = \left[PNG\left(N, \hat{\underline{\omega}}_{s_{ij}}\right)\right] + \left[T^b_f(q_i)\right]_i \left[\overline{N}\right] \hat{\underline{a}}_j$ $a^b_{x_{i_d}} = \frac{\delta T_i - \delta D_i}{m_i} = 0$	**Guidance Laws**
4	$\frac{d}{dt}\underline{a}^b_i = [-\Lambda_i]\,\underline{a}^b_i + [\Lambda_i]\,\underline{a}^b_{i_d}$	**Autopilot**

Table A.1: (continued) State Space Dynamics Model for Navigation, Seeker, Guidance and Autopilot.

	ALGORITHM	MODULE NAME
5	**5.1. Quaternions:** $q_{1_i} = \cos\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\cos\frac{\psi_i}{2} + \sin\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\sin\frac{\psi_i}{2}$ $q_{2_i} = \sin\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\cos\frac{\psi_i}{2} - \cos\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\sin\frac{\psi_i}{2}$ $q_{3_i} = \cos\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\cos\frac{\psi_i}{2} + \sin\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\sin\frac{\psi_i}{2}$ $q_{4_i} = \cos\frac{\phi_i}{2}\cos\frac{\theta_i}{2}\sin\frac{\psi_i}{2} + \sin\frac{\phi_i}{2}\sin\frac{\theta_i}{2}\cos\frac{\psi_i}{2}$ **5.2. Quaternion Evolution:** $\frac{d}{dt}\underline{q}_i = \left[\Omega_i^b\right]\underline{q}_i\ ;\ \underline{\omega}_i^b = \frac{\underline{u}_i^b \times \underline{a}_i^b}{\underline{u}_i^{b\,T}\,\underline{u}_i^b}$ **5.3. The Transformation Matrix:** $t_{11_i} = \left(q_{1_i}{}^2 + q_{2_i}{}^2 - q_{3_i}{}^2 - q_{4_i}{}^2\right)$ $t_{12_i} = 2\left(q_{2_i}q_{3_i} + q_{1_i}q_{4_i}\right)$ $t_{13_i} = 2\left(q_{2_i}q_{4_i} - q_{1_i}q_{3_i}\right)$ $t_{21_i} = 2\left(q_{2_i}q_{3_i} - q_{1_i}q_{4_i}\right)$ $t_{22_i} = \left(q_{1_i}{}^2 - q_{2_i}{}^2 + q_{3_i}{}^2 - q_{4_i}{}^2\right)$ $t_{23_i} = 2\left(q_{3_i}q_{4_i} + q_{1_i}q_{2_i}\right)$ $t_{31_i} = 2\left(q_{2_i}q_{4_i} + q_{1_i}q_{3_i}\right)$ $t_{32_i} = 2\left(q_{3_i}q_{4_i} - q_{1_i}q_{2_i}\right)$ $t_{33_i} = \left(q_{1_i}{}^2 - q_{2_i}{}^2 - q_{3_i}{}^2 + q_{4_i}{}^2\right)$ $\left[T_f^b\right]_i = \begin{bmatrix} t_{11_i} & t_{12_i} & t_{13_i} \\ t_{21_i} & t_{22_i} & t_{23_i} \\ t_{31_i} & t32 & t_{33_i} \end{bmatrix};\ \left[T_b^f\right]_i = \left[T_f^b\right]_i^T$ $\underline{a}_i = \left[T_b^f\right]_i\ \underline{a}_i^b$	**Navigation Model**

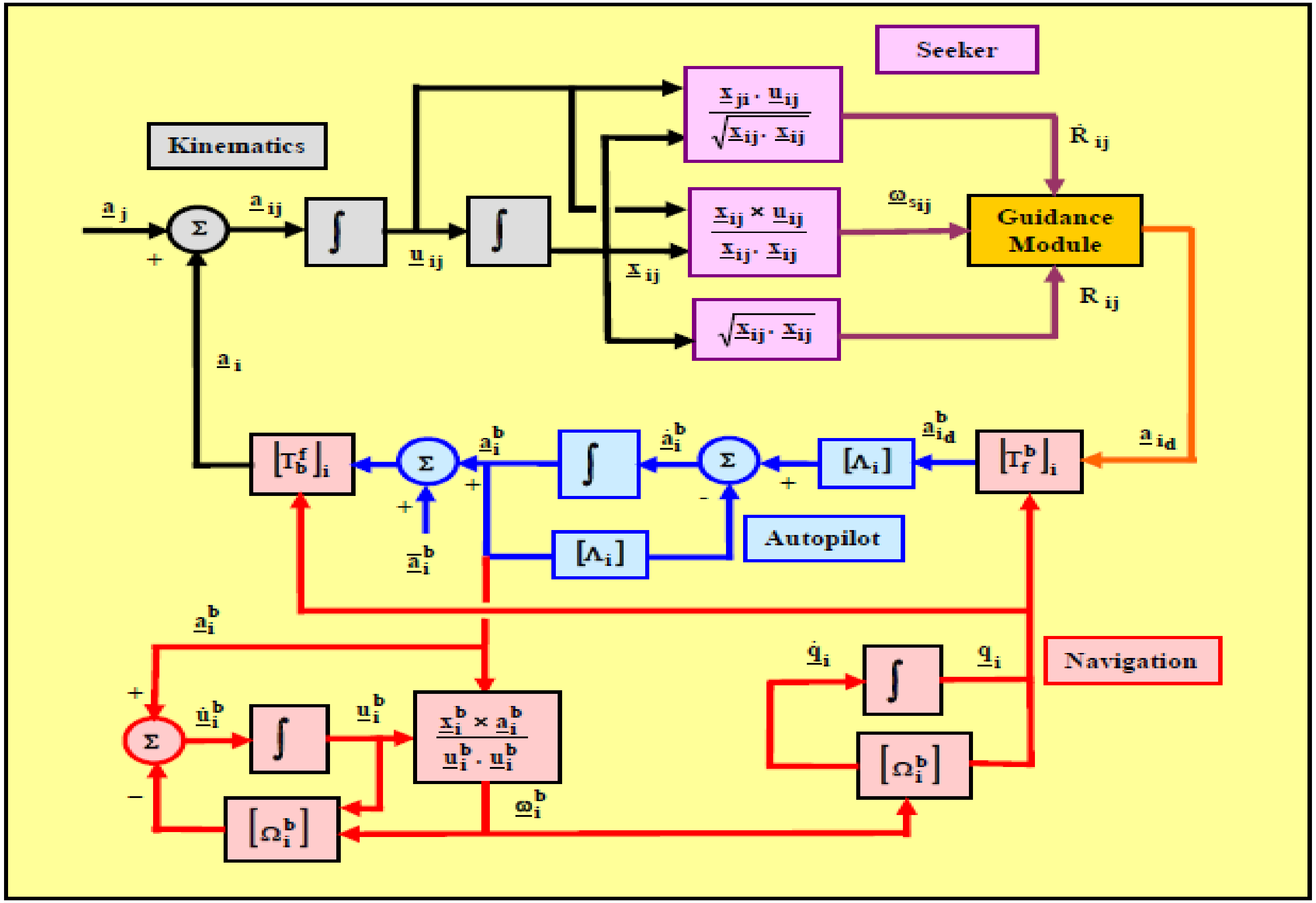

Figure A.1: Intelligent Integrated Navigation, Guidance & Control Test Bed

Appendix B

B.1 Aerodynamic Forces and Equations of Motion

For a symmetrical body $\left(\mathbf{I_{zx} = 0; I_y = I_z}\right)$, the equations of motion for an aerodynamic vehicle are given by (see Figure B.1) [4]:

$$\dot{\mathbf{u}}^{\mathbf{b}} + \mathbf{q}\mathbf{w}^{\mathbf{b}} - \mathbf{r}\mathbf{v}^{\mathbf{b}} = \frac{\mathbf{X}}{\mathbf{m}} - \mathbf{g}\sin\theta \qquad \text{(A-1.1)}$$

$$\dot{\mathbf{v}}^{\mathbf{b}} + \mathbf{r}\mathbf{u}^{\mathbf{b}} - \mathbf{p}\mathbf{w}^{\mathbf{b}} = \frac{\mathbf{Y}}{\mathbf{m}} + \mathbf{g}\cos\theta\sin\phi \qquad \text{(A-1.2)}$$

$$\dot{\mathbf{w}}^{\mathbf{b}} + \mathbf{p}\mathbf{v}^{\mathbf{b}} - \mathbf{q}\mathbf{u}^{\mathbf{b}} = \frac{\mathbf{Z}}{\mathbf{m}} + \mathbf{g}\cos\theta\cos\phi \qquad \text{(A-1.3)}$$

$$\dot{\mathbf{p}} + \mathbf{qr}\frac{\left(\mathbf{I_z} - \mathbf{I_y}\right)}{\mathbf{I_x}} = \frac{\mathbf{L}}{\mathbf{I_x}} \qquad \text{(A-1.4)}$$

$$\dot{\mathbf{q}} + \mathbf{rp}\frac{\left(\mathbf{I_x} - \mathbf{I_z}\right)}{\mathbf{I_y}} = \frac{\mathbf{M}}{\mathbf{I_y}} \qquad \text{(A-1.5)}$$

$$\dot{\mathbf{r}} + \mathbf{pq}\frac{\left(\mathbf{I_y} - \mathbf{I_x}\right)}{\mathbf{I_z}} = \frac{\mathbf{N}}{\mathbf{I_z}} \qquad \text{(A-1.6)}$$

Where:

$\left(\mathbf{u^b, v^b, w^b}\right)$**:** are the vehicle velocities defined in body axis.

$\left(\mathbf{p, q, r}\right)$**:** are the body rotation rates w.r.t the fixed axis defined in the body axis.

$\left(\mathbf{X, Y, Z}\right)$**:** are the aerodynamic forces acting on the vehicle body defined in the body axis.

$\left(\mathbf{L, M, N}\right)$**:** are the aerodynamic moments acting on the vehicle body defined in the body axis.

$\left(\mathbf{I_x, I_y, I_z}\right)$**:** are the vehicle body inertias.

$\mathbf{m}$ **:** is the vehicle mass.

$\left(\psi, \theta, \phi\right)$**:** are respectively the body (Euler) angles w.r.t the fixed axis.

For a non-rolling vehicle $\dot{\mathbf{p}} = \mathbf{p} = \phi = \mathbf{0}$; this assumption enables us to decouple the yaw and pitch kinematics. Equations (A-1.1)-(A1.6) give us:

$$\dot{\mathbf{u}}^{\mathbf{b}} + \mathbf{q}\mathbf{w}^{\mathbf{b}} - \mathbf{r}\mathbf{v}^{\mathbf{b}} = \frac{\mathbf{X}}{\mathbf{m}} - \mathbf{g}\sin\theta \qquad \text{(A-1.7)}$$

$$\dot{\mathbf{v}}^{\mathbf{b}} + \mathbf{r}\mathbf{u}^{\mathbf{b}} = \frac{\mathbf{Y}}{\mathbf{m}} \qquad \text{(A-1.8)}$$

$$\dot{\mathbf{w}}^{\mathbf{b}} - \mathbf{q}\mathbf{u}^{\mathbf{b}} = \frac{\mathbf{Z}}{\mathbf{m}} + \mathbf{g}\cos\theta \qquad \text{(A-1.9)}$$

$$\mathbf{L} = \mathbf{0} \qquad \text{(A-1.10)}$$

$$\dot{\mathbf{q}} = \frac{\mathbf{M}}{\mathbf{I_y}} \qquad \text{(A-1.11)}$$

$$\dot{\mathbf{r}} = \frac{\mathbf{N}}{\mathbf{I_z}} \qquad \text{(A-1.12)}$$

The accelerations about the vehicle body CG is given by:

$$\mathbf{a_x^b} = \mathbf{\dot{u}^b} + \mathbf{qw^b} - \mathbf{rv^b} = \frac{\mathbf{X}}{\mathbf{m}} - \mathbf{g}\sin\theta \quad \text{(A-1.13)}$$

$$\mathbf{a_y^b} = \mathbf{\dot{v}^b} + \mathbf{ru^b} = \frac{\mathbf{Y}}{\mathbf{m}} \quad \text{(A-1.14)}$$

$$\mathbf{a_z^b} = \mathbf{\dot{w}^b} - \mathbf{qu^b} = \frac{\mathbf{Z}}{\mathbf{m}} + \mathbf{g}\cos\theta \quad \text{(A-1.15)}$$

Where: $\left(\mathbf{a_x^b}, \mathbf{a_y^b}, \mathbf{a_z^b}\right)$**:** are the body accelerations w.r.t the fixed axis defined in the body axis.

If we consider perturbation about the nominal, we get:

$$\mathbf{\bar{a}_x^b} + \delta\mathbf{a_x^b} = \frac{\left(\mathbf{\bar{X}} + \delta\mathbf{X}\right)}{\mathbf{m}} - \mathbf{g}\left(\sin\bar{\theta} + \cos\bar{\theta}\,\delta\theta\right)$$

$$\mathbf{\bar{a}_y^b} + \delta\mathbf{a_y^b} = \frac{\left(\mathbf{\bar{Y}} + \delta\mathbf{Y}\right)}{\mathbf{m}}$$

$$\mathbf{\bar{a}_z^b} + \delta\mathbf{a_z^b} = \frac{\left(\mathbf{\bar{Z}} + \delta\mathbf{Z}\right)}{\mathbf{m}} + \mathbf{g}\left(\cos\bar{\theta} - \sin\bar{\theta}\,\delta\theta\right)$$

B.2 2-D Yaw-Plane Kinematics Equations:

For 2-D yaw-plane kinematics only $\bar{\theta} = 0; \delta\theta = 0$ (i.e. zero pitch motion), therefore, the X and Y-plane *steady state* equations (in body axis) may be written as:

$$\mathbf{\bar{a}_y^b} = \frac{\mathbf{\bar{Y}}}{\mathbf{m}} = \mathbf{\tilde{Y}} \quad \text{(A-1.16)}$$

$$\mathbf{\bar{a}_x^b} = \frac{\mathbf{\bar{X}}}{\mathbf{m}} = \frac{\left(\mathbf{\bar{T}} - \mathbf{\bar{D}}\right)}{\mathbf{m}} = \left(\mathbf{\tilde{T}} - \mathbf{\tilde{D}}\right) \quad \text{(A-1.17)}$$

Where we define: $\frac{\mathbf{\bar{Y}}}{\mathbf{m}} = \mathbf{\tilde{Y}}$; $\frac{\mathbf{\bar{X}}}{\mathbf{m}} = \frac{\left(\mathbf{\bar{T}} - \mathbf{\bar{D}}\right)}{\mathbf{m}} = \left(\mathbf{\tilde{T}} - \mathbf{\tilde{D}}\right)$. Also, the total thrust is defined as: $\mathbf{T} = \mathbf{\bar{T}} + \delta\mathbf{T}$, and the total drag is defined as: $\mathbf{D} = \mathbf{\bar{D}} + \delta\mathbf{D}$.

For 'nominal flight' condition in the yaw-plane $\mathbf{\bar{Y}} = 0$; and the perturbation equation is given by:

$$\delta\mathbf{a_y^b} = \frac{\delta\mathbf{Y}}{\mathbf{m}} = \delta\mathbf{\tilde{Y}} \quad \text{(A-1.18)}$$

$$\delta\mathbf{a_x^b} = \frac{\delta\mathbf{X}}{\mathbf{m}} = \frac{\left(\delta\mathbf{T} - \delta\mathbf{D}\right)}{\mathbf{m}} = \left(\delta\mathbf{\tilde{T}} - \delta\mathbf{\tilde{D}}\right) \quad \text{(A-1.19)}$$

Where :

$\delta\mathbf{a_y^b}$ **:** represents the body axis guidance commands (lateral acceleration) applied by the vehicle.

$\delta \mathbf{a_x^b}$ **:** represents the body axis guidance commands (longitudinal acceleration) applied by the vehicle. However, during guidance manoeuvre $(\delta\tilde{\mathbf{T}}, \delta\tilde{\mathbf{D}})$ are not directly controlled, Hence we may assume $\delta \mathbf{a_x^b}$ to be zero.

B.3 2-D Pitch-Plane Kinematics Equations:

Unlike the previous case, for 2-D pitch-plane kinematics $\overline{\theta} \neq \mathbf{0}$, and for steady state conditions, we get:

$$\overline{\mathbf{a}}_\mathbf{z}^\mathbf{b} = \frac{\overline{\mathbf{Z}}}{\mathbf{m}} + \mathbf{g}\cos\overline{\theta} = \tilde{\mathbf{Z}} + \mathbf{g}\cos\overline{\theta} \tag{A-1.20}$$

$$\overline{\mathbf{a}}_\mathbf{x}^\mathbf{b} = \frac{\overline{\mathbf{X}}}{\mathbf{m}} - \mathbf{g}\sin\overline{\theta} = \frac{(\overline{\mathbf{T}} - \overline{\mathbf{D}})}{\mathbf{m}} - \mathbf{g}\sin\overline{\theta} = (\tilde{\mathbf{T}} - \tilde{\mathbf{D}}) - \mathbf{g}\sin\overline{\theta} \tag{A-1.21}$$

The X , Z (pitch)-plane perturbation kinematics (in body axis) is given by:

$$\delta\mathbf{a}_\mathbf{z}^\mathbf{b} = \frac{\delta\mathbf{Z}}{\mathbf{m}} = \delta\tilde{\mathbf{Z}} \tag{A-1.22}$$

$$\delta\mathbf{a}_\mathbf{x}^\mathbf{b} = \frac{(\delta\mathbf{T} - \delta\mathbf{D})}{\mathbf{m}} = (\delta\tilde{\mathbf{T}} - \delta\tilde{\mathbf{D}}) \tag{A-1.23}$$

Where

$\delta \mathbf{a_z^b}$ **:** represents the body axis guidance commands (lateral acceleration) applied by the vehicle.

As in the case of the yaw-plane, during guidance $(\delta\tilde{\mathbf{T}}, \delta\tilde{\mathbf{D}})$ are not directly controlled, hence we may assume $\delta\mathbf{a_x^b}$ to be zero. The reader will recognize, that in the main text of this report:

$$\mathbf{a}^\mathbf{b}_{\mathbf{x}_{\mathbf{d_i}}} \equiv \delta\,\mathbf{a_x}\,,\; \mathbf{a}^\mathbf{b}_{\mathbf{y}_{\mathbf{d_i}}} \equiv \delta\,\mathbf{a_y}\;\; \mathbf{a}^\mathbf{b}_{\mathbf{z}_{\mathbf{d_i}}} \equiv \delta\,\mathbf{a_z} \tag{A-1.24}$$

B.4 Calculating the Aerodynamic Forces

For the purposes of the simulation under consideration we may assume that the vehicle thrust profile $\overline{\mathbf{T}}(\mathbf{t})$, say as a function of time, is given; then the drag force $\overline{\mathbf{D}}$, which depends on the vehicle aerodynamic configuration, is given by:

$$\overline{\mathbf{D}} = \left(\frac{\mathbf{1}}{\mathbf{2}}\rho\overline{\mathbf{V}}^\mathbf{2}\mathbf{S}\right)\mathbf{C_D}(\overline{\alpha}, \overline{\beta}) \tag{A1.25}$$

$$\overline{\mathbf{Y}} = \left(\frac{\mathbf{1}}{\mathbf{2}}\rho\overline{\mathbf{V}}^\mathbf{2}\mathbf{S}\right)\mathbf{C_L}(\overline{\beta}) = \mathbf{0} \tag{A1.26}$$

$$\overline{\mathbf{Z}} = \left(\frac{\mathbf{1}}{\mathbf{2}}\rho\overline{\mathbf{V}}^\mathbf{2}\mathbf{S}\right)\mathbf{C_L}(\overline{\alpha}) = -\mathbf{g}\cos\overline{\theta} \tag{A1.27}$$

Where: the term in the bracket is the dynamic pressure; ρ being the air density, $\mathbf{S}$ is the body characteristic surface area and $\overline{\mathbf{V}}$ is the steady-state velocity. $\mathbf{C_D}$ is the drag coefficient

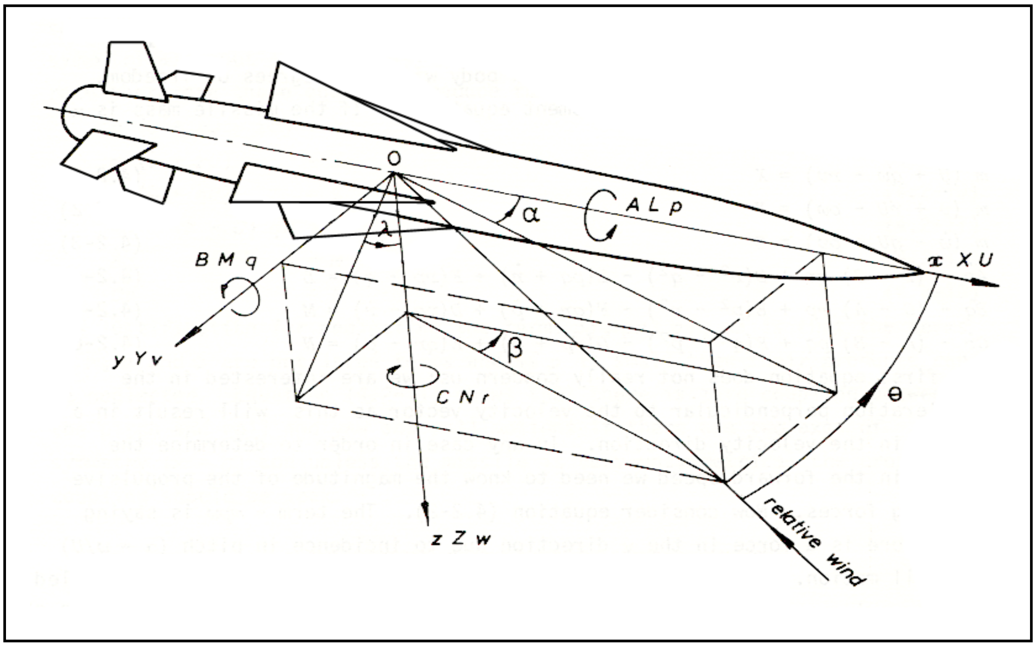

Figure B.1: Aerodynamic variables for a missile

and $\mathbf{C_L}$ is the lift coefficient. $(\bar{\alpha},\bar{\beta})$ represent respectively the pitch- and the yaw-plane nominal (steady-state) incidence angles. Contribution to thrust and drag due to control deflections are small and ignored. Also:

$$\delta\mathbf{Y} = \left(\frac{1}{2}\rho\mathbf{V}^2\mathbf{S}\right)\mathbf{C_L}(\delta\beta) = \mathbf{0} \tag{A1.28}$$

$$\delta\mathbf{Z} = \left(\frac{1}{2}\rho\mathbf{V}^2\mathbf{S}\right)\mathbf{C_L}(\delta\alpha) = -\mathbf{g}\sin\bar{\theta}\,\delta\theta \tag{A1.29}$$

$(\delta\alpha,\delta\beta)$ represent respectively the variation in pitch- and the yaw-plane incidence angles as a result of control demands; these are assumed to be small. Note that for a given $(\delta\alpha,\delta\beta)$, $\delta\mathbf{Y},\delta\mathbf{Z} \propto \mathbf{V}^2$, the maximum/minimum acceleration capability of a vehicle is rated at the nominal velocity $\bar{\mathbf{V}}$, then the maximum/minimum acceleration at any other velocity $\mathbf{V}$ is given by:

$$\left\|\mathbf{a}^{\mathbf{b}}_{\mathbf{y_d}}\right\| \le \mu\,\mathbf{a}^{\mathbf{b}}_{\mathbf{y_{max}}}\,;\ \left\|\mathbf{a}^{\mathbf{b}}_{\mathbf{z_d}}\right\| \le \mu\,\mathbf{a}^{\mathbf{b}}_{\mathbf{z_{max}}} \qquad ;\text{Where: } \mu = \left(\frac{\mathbf{V}}{\bar{\bar{\mathbf{V}}}}\right)^2$$

B.5 Body Incidence

The body incidence angles (α,β) are given by $(\mathbf{v_b},\mathbf{w_b} \ll \mathbf{u_b})$:

$$\alpha = \tan^{-1}\left(\frac{\mathbf{w^b}}{\mathbf{u^b}}\right) \approx \frac{\mathbf{w^b}}{\mathbf{u^b}}\,;\ \beta = \tan^{-1}\left(\frac{\mathbf{v^b}}{\mathbf{u^b}}\right) \approx \frac{\mathbf{v^b}}{\mathbf{u^b}}\,;\mathbf{V_b} = \mathbf{V_i} = \left(=\sqrt{\mathbf{u_b}^2 + \mathbf{v_b}^2 + \mathbf{w_b}^2}\right)$$; these angles represent the angle that the body makes w.r.t 'flight path' or with the direction of the

total velocity vector $\mathbf{V}$. In this report we shall assume that these angles are small and may be ignored; in which case the body can be assumed to be aligned with the velocity vector.

Appendix C 3-D Collision Course Engagement Geometry & Computing Missile Collision Course Heading

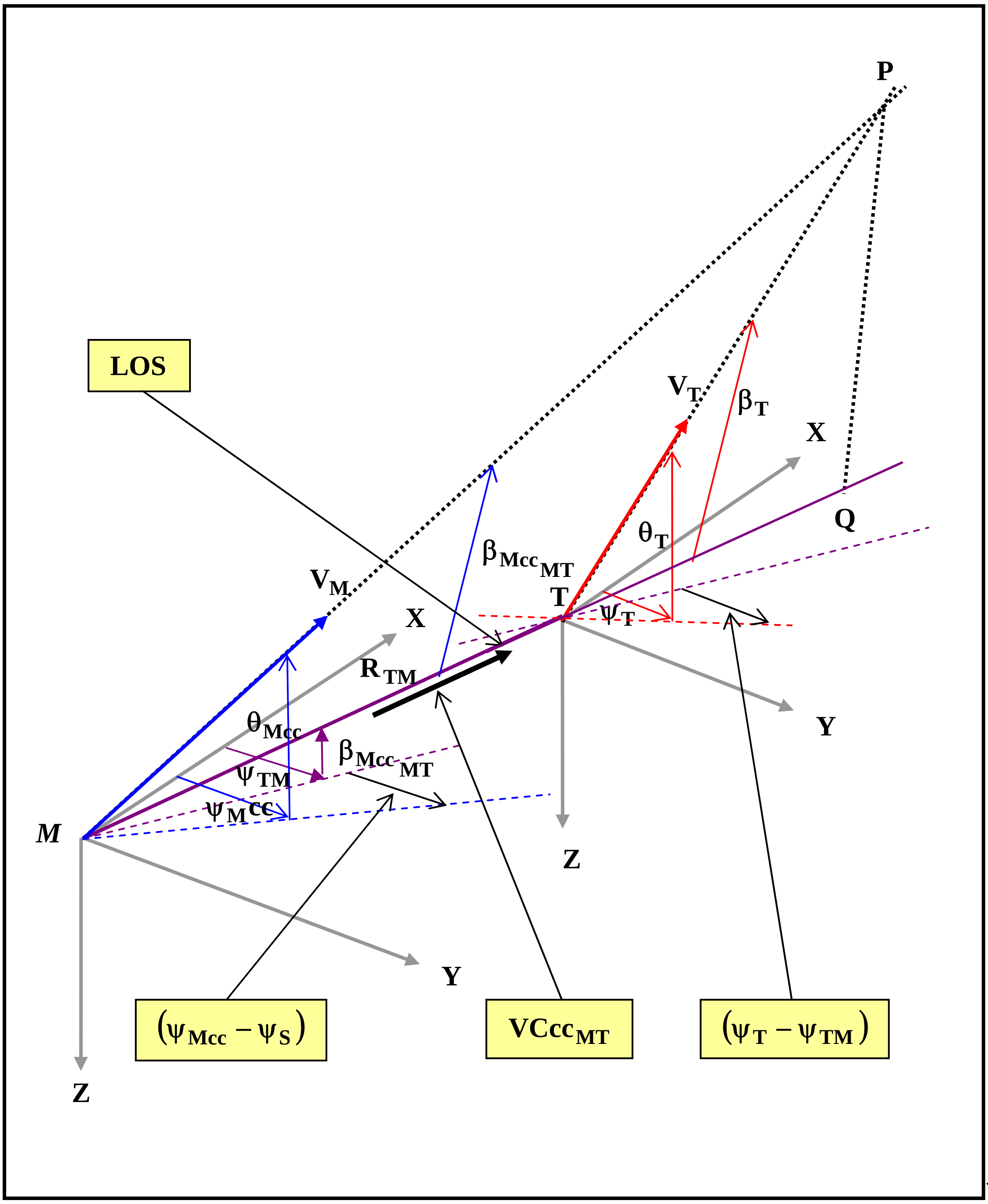

Figure C.1: Three-DOF Collision Course Engagement Geometry.

C.1 Computing 3-DOF Collision Course Missile Heading Angles:

C.1.1 Computing (β_{TM}), Given $(V_T, \theta_T, \psi_T, \theta_{TM}, \psi_{TM})$:

The unit vector along the target body $\underline{ev}_T$ and the unit LOS vector $\underline{es}_{TM}$ may be written as:

$$\underline{ev}_T = [\cos\theta_T \cos\psi_T \quad \cos\theta_T \sin\psi_T \quad \sin\theta_T] \qquad \text{(A2.1)}$$

$$\underline{es}_{TM} = [\cos\theta_{TM} \cos\psi_{TM} \quad \cos\theta_{TM} \sin\psi_{TM} \quad \sin\theta_{TM}] \qquad \text{(A2.2)}$$

It follows from equations (A2.1), (A2.2) that:

$$\begin{aligned}\langle \underline{ev}_T, \underline{es}_{TM} \rangle = \cos\beta_{TM} &= \cos\theta_T \cos\psi_T \cos\theta_{TM} \cos\psi_{TM} \\ &+ \cos\theta_T \sin\psi_T \cos\theta_{TM} \sin\psi_{TM} + \sin\theta_T \sin\theta_{TM}\end{aligned} \qquad \text{(A2.3)}$$

Equation (A2.3), gives us:

$$\beta_{TM} = \cos^{-1}\begin{Bmatrix}\cos\theta_T \cos\psi_T \cos\theta_{TM} \cos\psi_{TM} \\ + \cos\theta_T \sin\psi_T \cos\theta_{TM} \sin\psi_{TM} + \sin\theta_T \sin\theta_{TM}\end{Bmatrix} \qquad \text{(A2.4)}$$

Where:

β_{TM} **:** is the angle between the target velocity vector and the target-to-missile LOS vector measured in $(V_T \times R_{TM} \times V_M - \text{plane})$.

V_T **:** is the target velocity.

V_M **:** is the target velocity.

(θ_T, ψ_T)**:** are respectively the body pitch and yaw body angles (Euler angles) w.r.t the fixed axis.

(θ_{TM}, ψ_{TM})**:** are respectively the target-to-missile LOS elevation and azimuth w.r.t the fixed axis.

C.1.2 Computing (βcc_{MT}), Given (V_M, β_{TM}) :

Consideration of collision course engagement in $(V_M \times V_T \times R_{MT} - \text{plane})$ gives us:

$$V_M \sin\beta cc_{MT} = V_T \sin\beta_{TM} \qquad \text{(A2.5)}$$

Equation (A2.5) gives us:

$$\beta cc_{MT} = \sin^{-1}\left\{\frac{V_T}{V_M}\sin\beta_{TM}\right\} \qquad \text{(A2.6)}$$

Where:

βcc_{MT} **:** is the angle between the missile collision course velocity vector and the target-to-missile LOS vector measured in $(V_T \times R_{TM} \times V_M - \text{plane})$.

C.1.3 Computing the Closing Velocity $(VC_{CC_{MT}})$ and Time-to-go (T_{go}) :

We shall define:

$$\mathbf{VCcc_{MT} = V_M \cos\beta cc_{MT} - V_T \cos\beta_{TM}} \tag{A2.7}$$

Also, the target/missile range-to-go $(\mathbf{R_{TM}})$ is defined as:

$$\mathbf{R_{MT}} = \left[(\mathbf{X_M - X_T})^2 + (\mathbf{Y_M - Y_T})^2 + (\mathbf{Z_M - Z_T})^2 \right]^{\frac{1}{2}} \tag{A2.8}$$

Then:

$$\mathbf{T_{go}} = \frac{\mathbf{R_{MT}}}{\mathbf{VCcc_{MT}}} \tag{A2.9}$$

Where:

$\mathbf{VCcc_{MT}}$ **:** is the collision course relative (closing) velocity of the missile w.r.t. the target. Note that the collision course velocity is along the range vector $\mathbf{R_{MT}}$.

$\mathbf{T_{go}}$ **:** is time-to-go (to intercept).

2.1.4. Computing $(\theta_{M_{CC}}, \psi_{M_{CC}})$:

Now the components of the relative position vector of the missile w.r.t may be written as :

$$\begin{aligned}\mathbf{X_M - X_T} &= (\mathbf{VCcc_{MT} \cos\theta_{TM} \cos\psi_{TM}}).\mathbf{T_{go}} \\ &= (\mathbf{V_M \cos\theta cc_M \cos\psi cc_M - V_T \cos\theta_T \cos\psi_T}).\mathbf{T_{go}}\end{aligned} \tag{A2.10}$$

$$\begin{aligned}\mathbf{Y_M - Y_T} &= (\mathbf{VCcc_{MT} \cos\theta_{TM} \sin\psi_{TM}}).\mathbf{T_{go}} \\ &= (\mathbf{V_M \cos\theta cc_M \sin\psi cc_M - V_T \cos\theta_T \sin\psi_T}).\mathbf{T_{go}}\end{aligned} \tag{A2.11}$$

$$\begin{aligned}\mathbf{Z_M - Z_T} &= (\mathbf{VC_{CC_{MT}} \sin\theta_{TM}}).\mathbf{T_{go}} \\ &= (\mathbf{V_M \sin\theta cc_M - V_T \sin\theta_T}).\mathbf{T_{go}}\end{aligned} \tag{A2.12}$$

Where:

$\psi_{M_{CC}}$ **:** is the missile collision course azimuth heading, measured w.r.t the fixed axis.

$\theta_{M_{CC}}$ **:** is the missile collision course elevation heading, measured w.r.t the fixed axis.

Equations (A2.10), (A2.11) and (A2.12) give us:

$$\begin{aligned}&\mathbf{\cos\theta cc_{MT} \cos\psi cc_{MT}} \\ &= \left(\frac{\mathbf{VC_{CC_{MT}}}}{\mathbf{V_M}} \mathbf{\cos\theta_{TM} \cos\psi_{TM}} + \frac{\mathbf{V_T}}{\mathbf{V_M}} \mathbf{\cos\theta_T \cos\psi_T} \right)\end{aligned} \tag{A2.13}$$

$$\begin{aligned}&\mathbf{\cos\theta cc_{MT} \sin\psi cc_{MT}} \\ &= \left(\frac{\mathbf{VC_{CC_{MT}}}}{\mathbf{V_M}} \mathbf{\cos\theta_{TM} \sin\psi_{TM}} + \frac{\mathbf{V_T}}{\mathbf{V_M}} \mathbf{\cos\theta_T \sin\psi_T} \right)\end{aligned} \tag{A2.14}$$

$$\sin\theta cc_{MT} = \left(\frac{VC_{CC_{MT}}}{V_M}\sin\theta_{TM} + \frac{V_T}{V_M}\sin\theta_T\right) \quad \text{(A2.15)}$$

Equation (A2.15) gives us:

$$\theta cc_{MT} = \sin^{-1}\left(\frac{VC_{CC_{MT}}}{V_M}\sin\theta_{TM} + \frac{V_T}{V_M}\sin\theta_T\right) \quad \text{(A2.16)}$$

Similarly equations (A2.13), (A2.14) (after some straight forward algebraic manipulation give us:

$$\psi cc_{MT} = \tan^{-1}\left\{\frac{\left(\frac{VC_{CC_{MT}}}{V_M}\cos\theta_{TM}\sin\psi_{TM} + \frac{V_T}{V_M}\cos\theta_T\sin\psi_T\right)}{\left(\frac{VC_{CC_{MT}}}{V_M}\cos\theta_{TM}\sin\psi_{TM} + \frac{V_T}{V_M}\cos\theta_T\sin\psi_T\right)}\right\} \quad \text{(A2.17)}$$

C.2 Computing 2-DOF Collision Course Missile Heading Angles:

C.2.1 Vertical Plane $(\mathbf{X}\times\mathbf{Z}-\mathbf{plane})$ Engagement:

For this case $(\psi_T = \psi_{Mcc} = \psi_S = 0)$ and $(\beta_T = (\theta_T - \theta_S))$, $(\beta_{Mcc} = (\theta_{Mcc} - \theta_S))$; hence utilising equation (A2.5) we get:

$$(\theta_{Mcc} - \theta_S) = \sin^{-1}\left\{\frac{V_T}{V_M}\sin(\theta_T - \theta_S)\right\} \quad \text{(A2.18)}$$

➔

$$\theta_{Mcc} = \theta_S + \sin^{-1}\left\{\frac{V_T}{V_M}\sin(\theta_T - \theta_S)\right\} \quad \text{(A2.19)}$$

C.2.2 Horizontal Plane $(\mathbf{XxY}-\mathbf{plane})$ Engagement:

For this case $(\theta_T = \theta_{Mcc} = \theta_S = 0)$ and $(\beta_T = (\psi_T - \psi_S))$, $(\beta_{Mcc} = (\psi_{Mcc} - \psi_S))$; hence utilising, as previously, equation (5) we get:

$$\psi_{Mcc} = \psi_S + \sin^{-1}\left\{\frac{V_T}{V_M}\sin(\psi_T - \psi_S)\right\} \quad \text{(A2.20)}$$

Page classification: UNCLASSIFIED

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION DOCUMENT CONTROL DATA	1. PRIVACY MARKING/CAVEAT (OF DOCUMENT)

2. TITLE	3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION)
Integrated Navigation, Guidance, and Control of Missile Systems: 3-D Dynamic Model	Document (U) Title (U) Abstract (U)
4. AUTHOR(S)	5. CORPORATE AUTHOR
Farhan A. Faruqi	DSTO Defence Science and Technology Organisation PO Box 1500 Edinburgh South Australia 5111 Australia

6a. DSTO NUMBER	6b. AR NUMBER	6c. TYPE OF REPORT	7. DOCUMENT DATE
DSTO-TR-2805	AR-015-533	Technical Report	February 2013

8. FILE NUMBER	9. TASK NUMBER	10. TASK SPONSOR	11. NO. OF PAGES	12. NO. OF REFERENCES
-	-	-	27	5

13. DSTO Publications Repository	14. RELEASE AUTHORITY
http://dspace.dsto.defence.gov.au/dspace/	Chief, Weapons Systems Division

15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT

Approved for public release

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18. DSTO RESEARCH LIBRARY THESAURUS

Modelling; Missile Guidance; Missile control

19. ABSTRACT

In this report a 3-D mathematical modelfor multi-party engagement kinematics is derived suitable for developing, implementing and testing modern missile guidance systems. The model developed here is suitable for both conventional and more advanced optimal intelligent guidance schemes including those that arise out of the differential game theory. This model accommodates changes in vehicle body attitude and other non-linear effects such as limits on lateral acceleration and aerodynamic forces.

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